2004年度 数学概論

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Introduction to Mathematics

Kazuaki TAIRA

Why do you study Mathematics ?

The Role of Mathematics

in

Natural Sciences

Mechanism of Mathematical Analysis

Natural Phenomenon

Mathematical Analysis

Mathematical Modeling

Differential Equations ⇒ Solution

Weather Forecast

Mechanism of Weather Forecast

Weather

Weather Forecast

Mathematical Modeling

Navier - Stokes Equations $\underset{\text{Numerical Analysis}}{\Rightarrow}$ Approximation Solution

Navier-Stokes Equations
in
Fluid Dynamics

$$\rho \frac{D\mathbf{V}}{Dt}$$

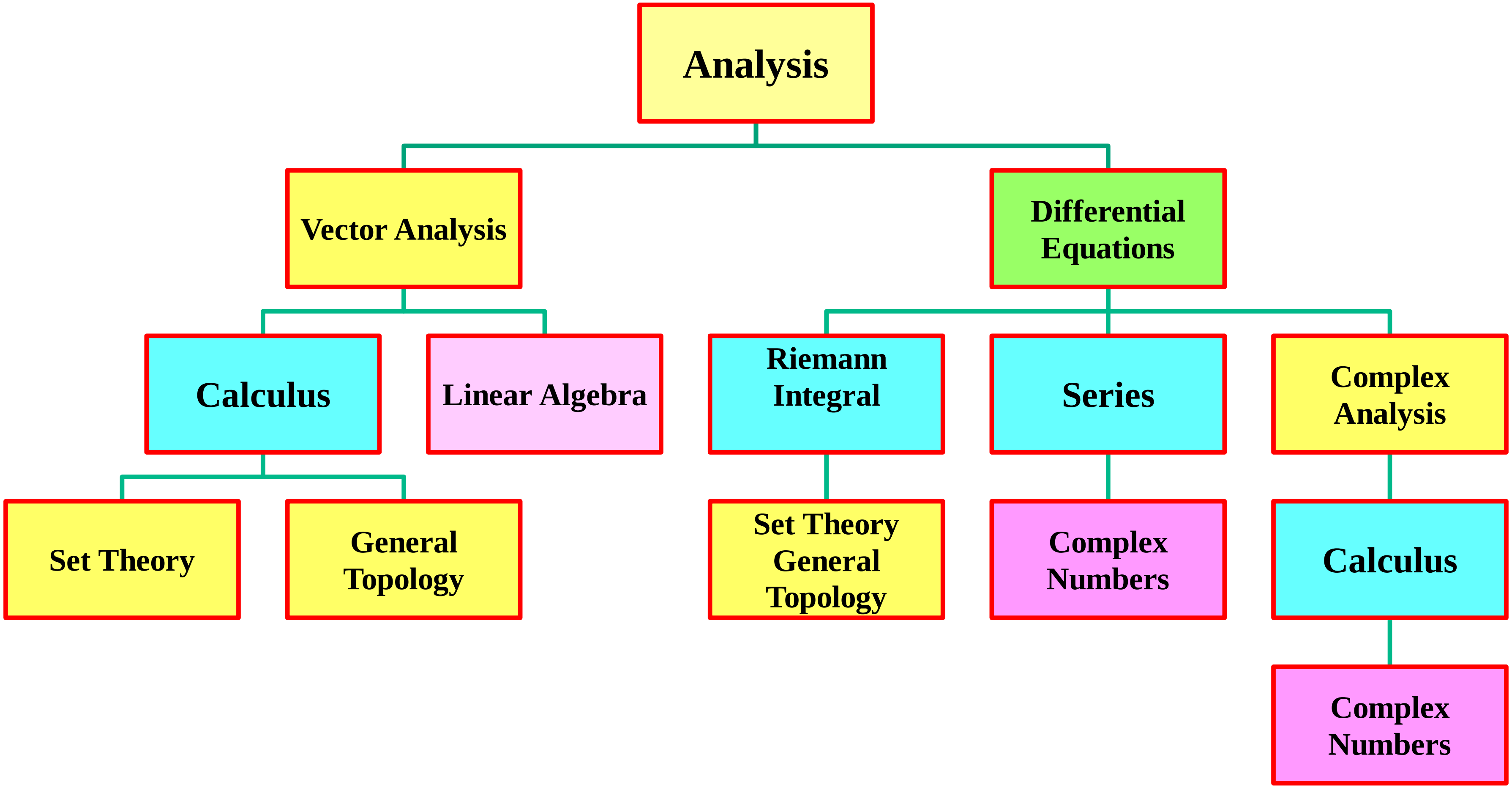
$$= -\nabla p + \rho\mathbf{B} + \mu\Delta\mathbf{V} + \frac{1}{3}\mu\nabla \cdot \mathrm{div}\,\mathbf{V}$$

Inertia Force

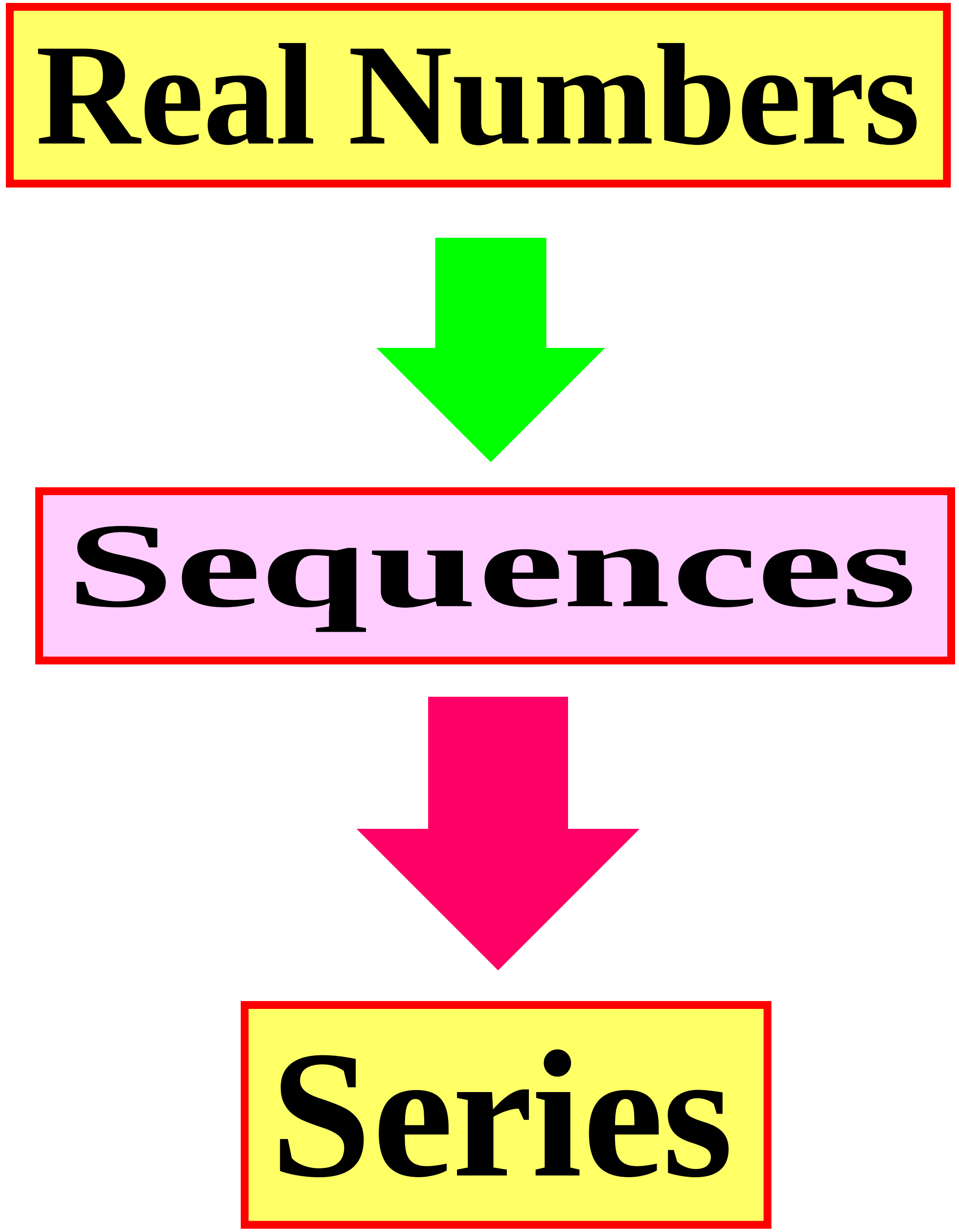
= Pressure + Force + Viscosity + Stress

Bird's-Eye View

Bird's- Eye View



Bird's-Eye View of Calculus


Real Numbers
Sequences
Series

Sequences

$\Downarrow$

Differentiation

$\Downarrow$

Differential Equations

Series

Integrals

Vector Analysis

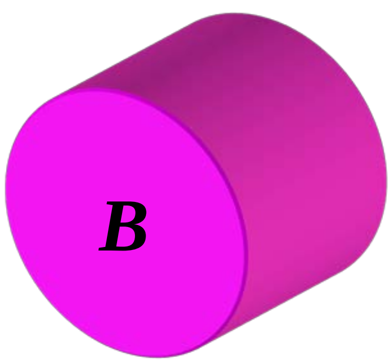
Mathematics versus Physics

Bird's-Eye View

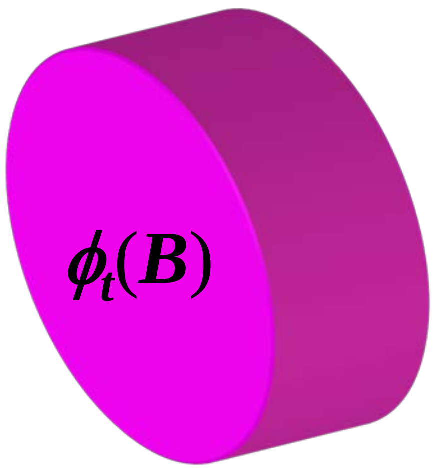
Theme	Mathematics	Physics
Differential Equations	Ordinary Differential Equations	Newton's Equation of Motion
Infinite Series	Fourier Series	Eigenfunction Expansions （Principle of Superposition）
Vector Analysis	Calculus on Surfaces	Continuum Mechanics

Mathematical Theory of Elasticity

Motions and Configurations



$$x = \phi_t\left(X\right)$$

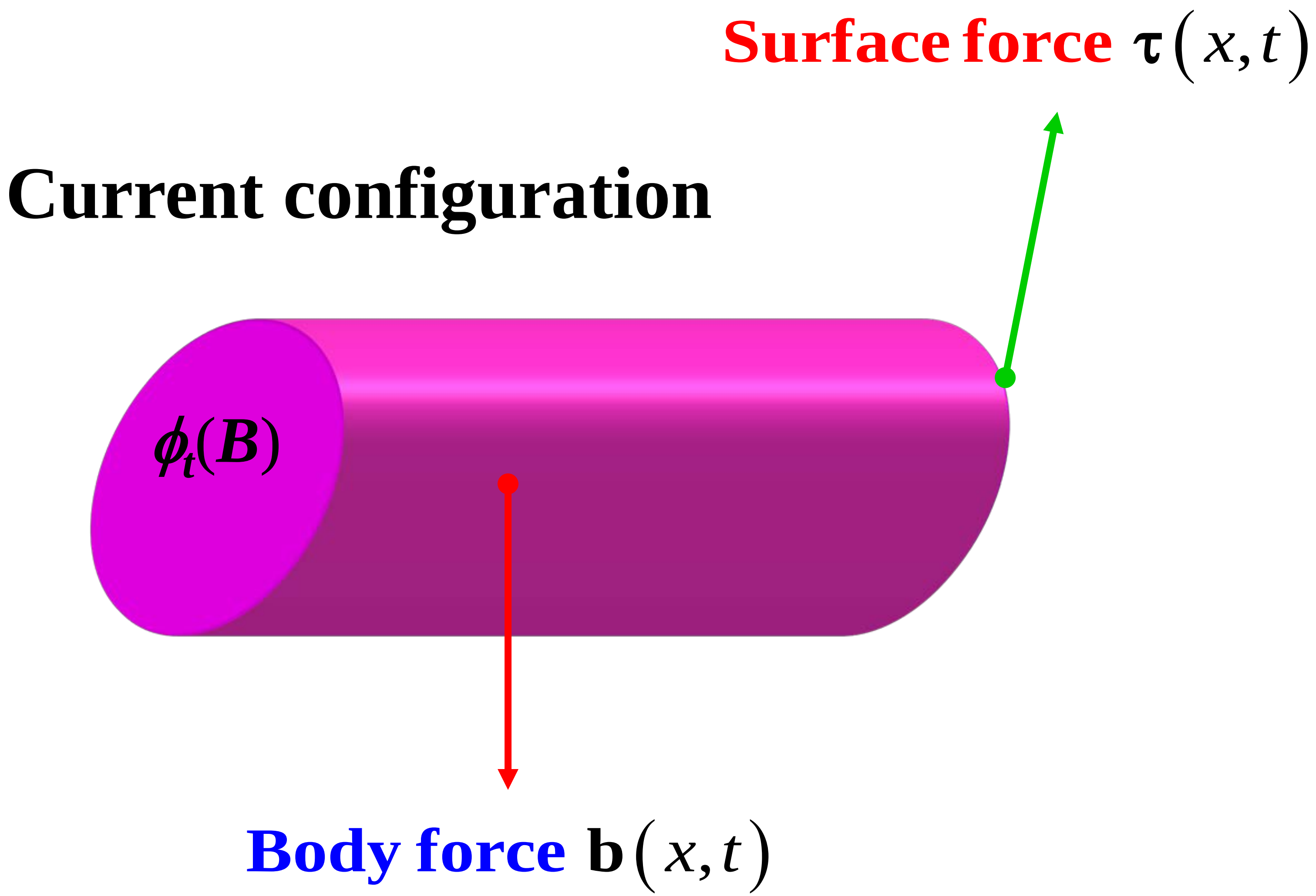


Reference configuration of a body

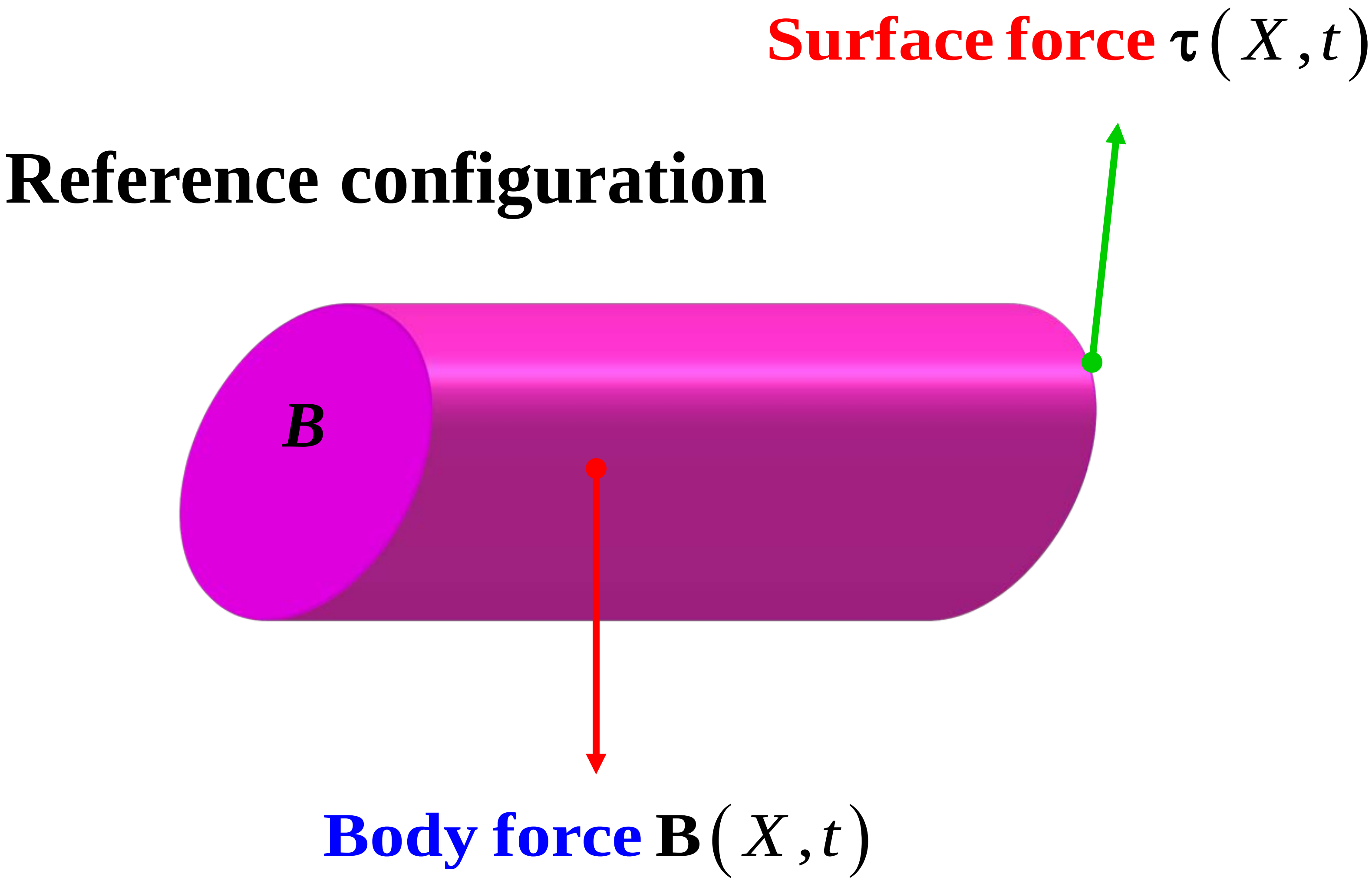
Body after time *t*

Two Descriptions in Elastodynamics

Euler's Description



Lagrange's Description



Continuum Mechanics (1)

Description	Conservation Law of Mass	Balance Law of Momentum
Euler	$\dot{\rho} + \rho \,\mathrm{div}\, \mathbf{v} = 0$	$\rho \dot{\mathbf{v}} = \mathrm{div}\, \boldsymbol{\sigma} + \rho \mathbf{b}$
Lagrange	$\rho_0(X) = \rho(\phi_t(X), t) J(X, t)$	$\rho_0 \dfrac{\partial \mathbf{V}}{\partial t} = \mathrm{Div}\, \mathbf{P} + \rho_0 \mathbf{B}$

Continuum Mechanics (2)

Description	Balance Law of Angular Momentum	Balance Law of Energy
Euler	$\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma}$	$\rho \dot{e} + \mathrm{div}\,\mathbf{q} = \mathrm{tr}(\boldsymbol{\sigma}\mathbf{d}) + \rho r$
Lagrange	$\mathbf{S} = {}^t\mathbf{S}$	$\rho_0 \dfrac{\partial E}{\partial t} + \mathrm{Div}\,\mathbf{Q} = \mathrm{tr}(\mathbf{S}\mathbf{D}) + \rho_0 R$

List of Mathematicians

List (1)

- **Archimedes**（B．C．287－B．C．212）**Greece**
- **Newton**（1642－1727）**England**
- **Leibniz**（1646－1716）**Germany**
- **Machin**（1685－1751）**England**
- **Fourier**（1736－1813）**France**
- **Lagrange**（1736－1813）**Italy, France**
- **Gauss**（1777－1855）**Germany**
- **Cauchy**（1789－1857）**France**
- **Abel**（1802－1829）**Norway**

List (2)

- Taylor（1685－1731）England
- Bolzano（1781－1848）Italy
- Hermite（1822－1901）France
- Maclaurin（1698－1746）Scotland
- Borel（1871－1956）France
- Dirichlet（1805－1859）Germany
- Weierstrass（1815－1897）Germany
- Dedekind（1831－1916）Germany

List (3)

- Rolle（1652－1719） France
- Laplace（1749－1827）France
- Riemann（1826－1866）Germany
- Hilbert（1862－1943） Germany
- Hadamard（1865－1963） France
- Lebesgue（1875－1941） France
- Euler（1707－1783）Switzerland
- Poincare（1854－1912） France

List (4)

- **Bernouille**（1667－1748）**Switzerland**
- **Bessel**（1784－1846）**Germany**
- **Cantor**（1845－1918) **Denmark/ Germany**
- **D'Alembert**（1717－1783）**France**
- **Darboux**（1842－1917） **France**
- **De Morgan**（1806－1871） **France**
- **Fubini**（1879－1943）**Italy**
- **de L'Hospital**（1661－1704） **France**

List (5)

- Stokes（1819－1903）England
- Stirling（1962－1770）
- Simpson（1710－1761）England
- Schwarz（1843－1921）Germany
- Peano（1858－1932）Italy
- Napier（1550－1617）Scotland
- Jordan（1838－1922）France
- Landau（1887－1938）

Mathematical Thoughts

Mathematical Thoughts

(Ⅰ) Mathematical Reasoning
(Ⅱ) Mathematical Ideas
(Ⅲ) Mathematical Image

Numerical Analysis

Role of Numerical Analysis

Mathematics	**Analysis**	**Numerical Analysis**
Physics	**Theoretical Physics**	**Physical Experiments**

Mathematics versus Physics

Bird's-Eye View

Theme	Mathematics	Physics
Differential Equations	Ordinary Differential Equations	Newton's Equation of Motion
Infinite Series	Fourier Series	Eigenfunction Expansions （Principle of Superposition）
Vector Analysis	Calculus on Surfaces	Continuum Mechanics

Elasticity

Importance of Elasticity

A human body is an elastic material

Thoughts and Methods

in

Analysis

Four Thoughts in Analysis

(Ⅰ) Discrete Case and Continuous Case

(Ⅱ) Principle of Superposition

(Ⅲ) Completeness

(Ⅳ) Numerical Analysis

Discrete Case
versus
Continuous Case

Vectors and Functions

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (\textbf{Finite - Dimensional Case})$$

Discrete Case

Continuous Case

$$\int_a^b K(t,s)x(s)ds = y(t)$$

(Infinite - Diemnsional Case)

Principle of Superposition

Principle of Superposition

Theme	Mathematics	Kinetics
Infinite Series	Fourier Series	Eigenfunction Expansions

Principle of Superposition

$$Pu = f, \quad u = \sum_i u_i$$

Decomposition into Fundamental Elements

Superposition of Solutions

$$f = \sum_i f_i$$

Find a solution

$$Pu_i = f_i$$

Jean Baptiste Joseph Fourier

Fourier

- **Jean Baptiste Joseph Fourier (1768-1830)**

French Mathematician and Physicist

La theorie analytique de la chaleur (1822)

Fourier's These

Every function of period 2π can be approximated in terms of trigonometric functions.

Fourier Series Expansion (1)

$$f(x) = \sum_{j=0}^{\infty} f_j(x)$$

$$= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

$$+ a_2 \cos 2x + b_2 \sin 2x + \cdots$$

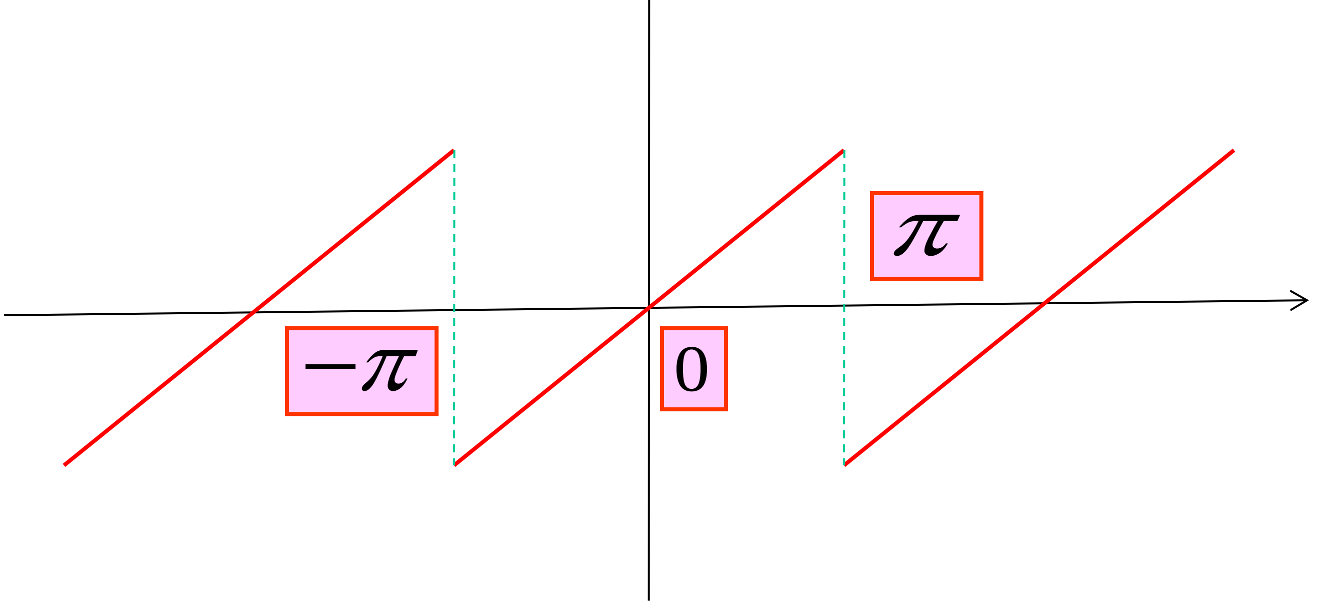
$$+ a_j \cos jx + b_j \sin jx + \cdots$$

Fourier Series Expansion (2)

$$\boxed{\begin{aligned} a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jx\,dx \\ b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jx\,dx \end{aligned}}$$

Example

$$\boxed{f(x) = x, \qquad -\pi < x < \pi}$$



Fourier Coefficients

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos jx\,dx = 0$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin jx\,dx = \frac{2}{j}(-1)^{j+1}$$

$$(j \neq 0)$$

Example of a Fourier Series

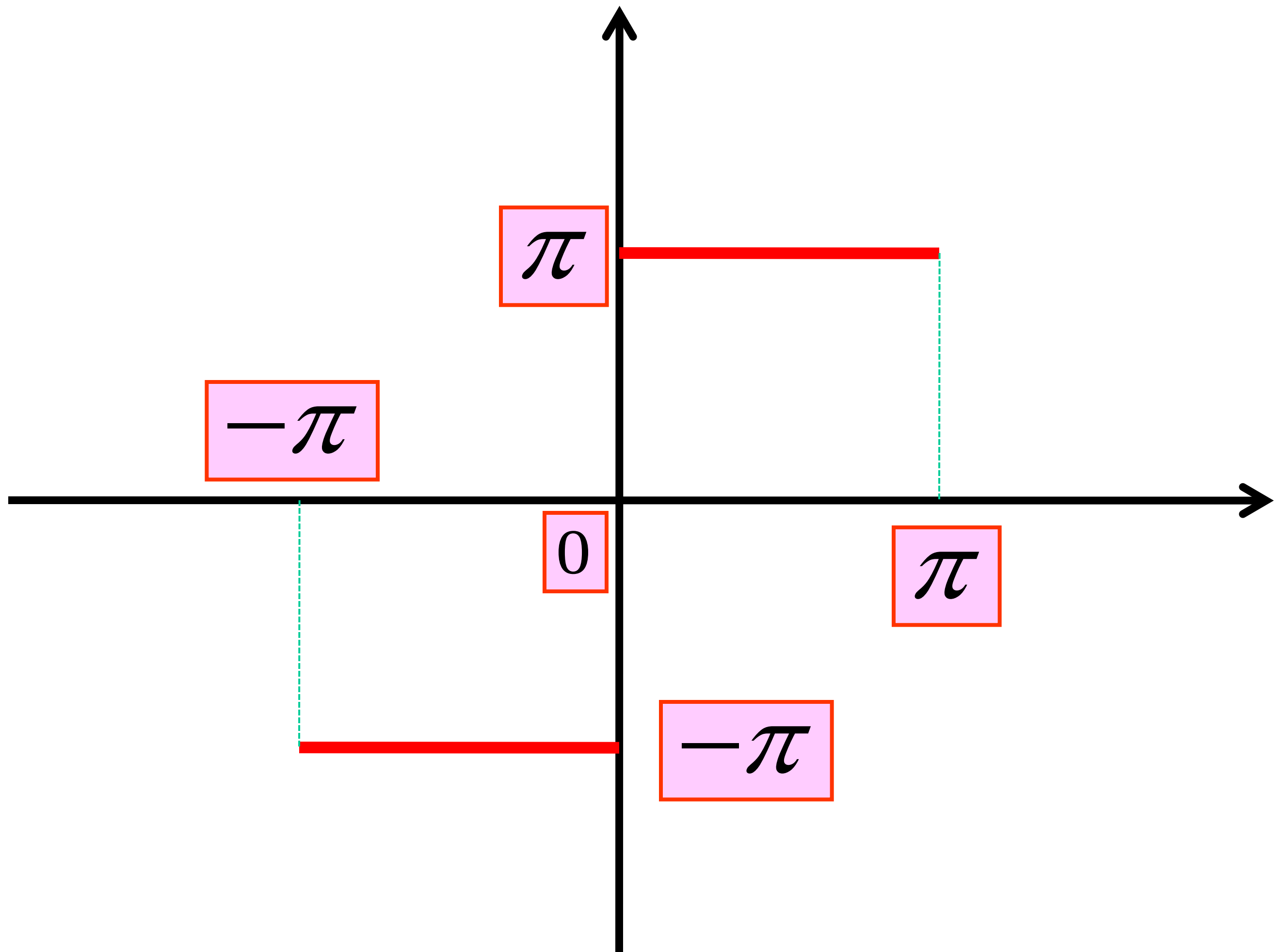
$$x = 2\sin x - 1\sin 2x + \cdots$$

$$+ \frac{2}{j}(-1)^{j+1}\sin jx + \cdots$$

$$(-\pi < x < \pi)$$

Fourier Series of Step Functions

Example of Step Functions



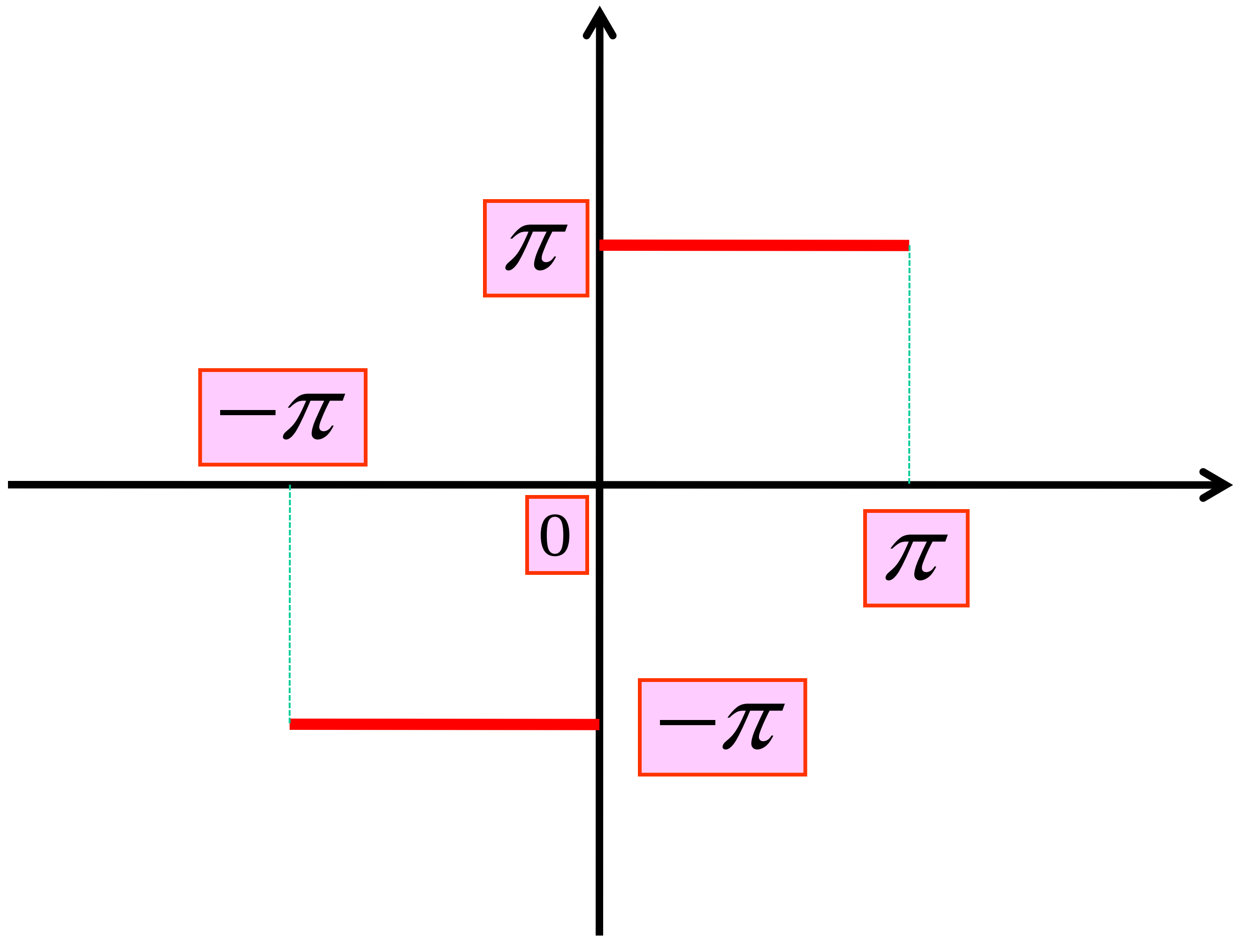
Example of Fourier Series

$$\sum_{j=0}^{\infty} \frac{1}{2j-1} \sin(2j-1)x = \begin{cases} \dfrac{\pi}{4} & 0 < x < \pi \\ 0 & x = 0, \pi \\ -\dfrac{\pi}{4} & -\pi < x < 0 \end{cases}$$

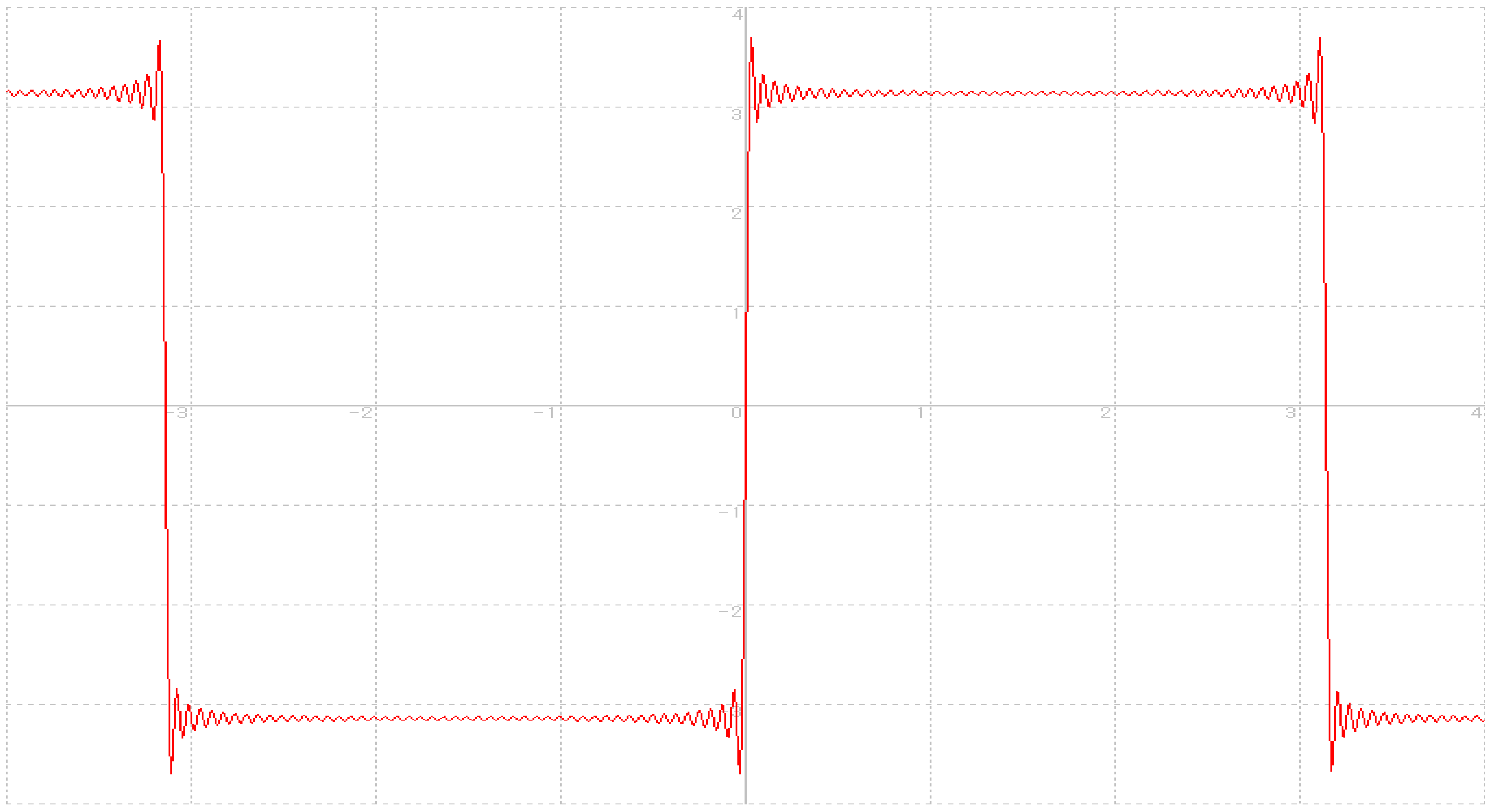
Gibbs Phenomenon

Numerical Computing with BASIC

Example of Step Functions



Example of Gibbs Phenomenon



Weierstrass' Continuous Function

Weierstrass's Function

$$f(x) = \sum_{k=0}^{\infty} a^k \cos\left(b^k x\right)$$

$$0 < a < 1, \quad ab \geq 1$$

Numerical Computing with BASIC

Example

$$f(x)=\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}\cos\left(3^{k}x\right)$$

$$a=\frac{1}{2},\, b=3\Rightarrow ab=\frac{3}{2}>1$$

$$s_0(x) = \cos x$$

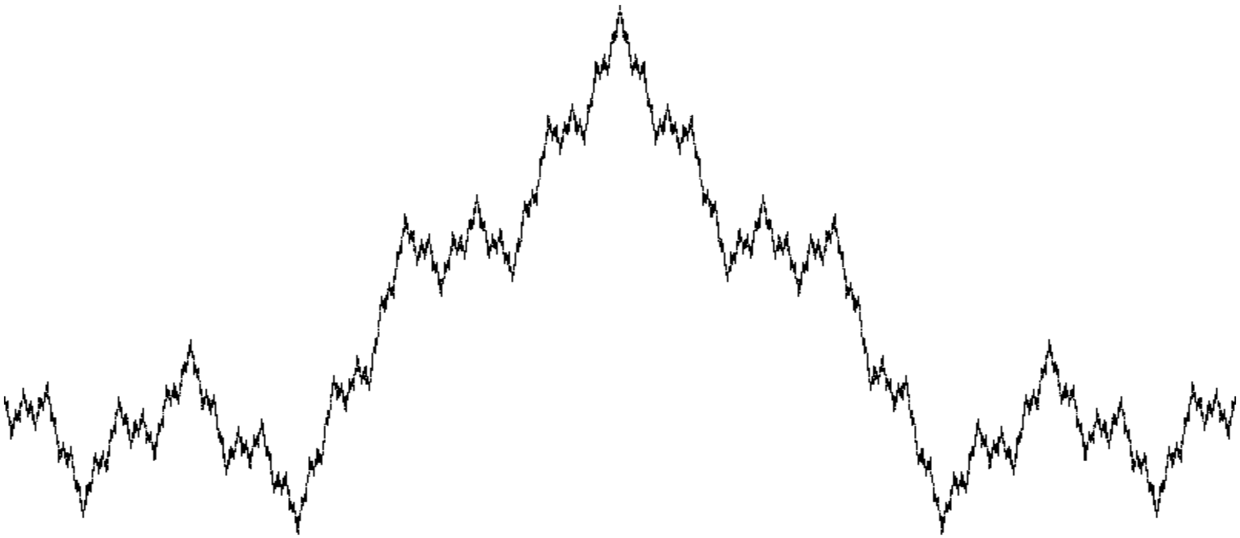
$$s_1(x) = \cos x + \frac{1}{2}\cos 3x$$

$$s_2(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x$$

$$s_3(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x$$

$$s_4(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x + \frac{1}{16}\cos 81x$$

Weierstrass Function



Heat Conduction (Fourier’s Work)

Formulation of a Problem

Steel bar of length π

Zero temperature on its ends

Initial temperature $f(x)$

Initial-Boundary Value Problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \ t > 0$$

$u(0,t) = u(\pi,t) = 0, \quad t > 0$ （**Boundary Condition**）

$u(x,0) = f(x), \quad 0 < x < \pi$ （**Initial Condition**）

Fourier’s Method
（Separation of Variables）

Representation of a Solution (Heat Kernel)

$$u(x,t) = \int_0^{\pi} p(t,x,y)\, f(y)\, dy$$

$$p(t,x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

(Heat Kernel)

Application to Series

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}}$$

Trace of a Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix}$$

$$\Rightarrow$$

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$ （**Sum of Eigenvalues**）

Trace Formula (1)

$$\int_0^{\pi} p(t,x,x)\,dx$$

$$=\frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\left(\int_0^{\pi}\sin^2 nx\,dx\right)$$

$$=\sum_{n=1}^{\infty} e^{-n^2 t}$$

Stationary Boundary Value Problem

$$v''(x) = g(x), \quad 0 < x < \pi$$

$$v(0) = v(\pi) = 0$$ （**Boundary Condition**）

Representation of a Solution (Green's Function)

$$\boxed{u(x,t) = \int_0^{\pi} G(x,y)g(y)dy}$$

$G(x,y)$ **Green Function**

Green's Function (Series Version)

$$G(x,y) = -\int_0^\infty p(t,x,y)\,dt$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^\infty e^{-n^2 t}\,dt\right)\sin nx \sin ny$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin nx \sin ny$$

Trace Formula (2)

$$\int_0^{\pi} G(x,x)\,dx = -\int_0^{\infty}\int_0^{\pi} p(t,x,x)\,dx\,dt$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int_0^{\pi}\sin^2 nx\,dx\right)$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n^2}$$ (**Sum of Eigenvalues**)

Green's Function (Integral Kernel Version)

$$G(x,y)=\begin{cases}\left(\dfrac{y}{\pi}-1\right)x & 0\le x\le y\le \pi\\[2ex] \left(\dfrac{x}{\pi}-1\right)y & 0\le y\le x\le \pi\end{cases}$$

Trace Formula (3)

$$\int_0^{\pi} G(x,x)\,dx$$

$$= \int_0^{\pi} \left(\frac{x^2}{\pi} - x \right) dx = -\frac{\pi^2}{6}$$

Trace Formula (4)

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = -\int_0^{\pi} G(x,x)\,dx = \frac{\pi^2}{6}}$$

Mathematical System of Numbers

Set	Algebra	Analysis
Complex Numbers	$+ - \times \div$	Complete
Real Numbers	$+ - \times \div$	Complete
Rational Numbers	$+ - \times \div$	
Integers	$+ - \times$	
Natural Numbers	$+ \times$	

Real Numbers

Real Numbers

Sequences

Series

Sequences

Differentiation

Differential Equations

Series

Integrals

Vector Analysis

Real Numbers and Decimal System

Real Numbers	Decimal System	Classification
Natural Numbers	Positive Integers	Rational
Integers	Integers	Rational
Fractional Numbers	Finite Decimal	Rational
Fractional Numbers	Recurring Decimal	Rational
Non-Fractional Numbers	Non-Recurring Decimal	Irrational

Finite Decimal (1)

$$\frac{1}{4} = 0.25$$

$$\frac{118}{25} = 4.72$$

Finite Decimal (2)

$$0.0625 = \frac{625}{10000}$$

$$= \frac{1}{16}$$

Recurring Decimal (1)

$$\frac{83}{74} = 1.12162162162216\cdots$$

$$= 1.1\dot{2}\dot{1}\dot{6}$$

$$\frac{89}{13} = 6.846153846153\cdots$$

$$= 6.\dot{8}\dot{4}\dot{6}\dot{1}\dot{5}\dot{3}$$

Recurring Decimal (2)

$$1.1\dot{2}1\dot{6} = 1.1216216216\cdots$$

$$= 1.1 + 0.0216 + 0.0000216 + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} + 216 \times \frac{1}{10^7} + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4}\left(1 + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} \times \frac{1}{1 - \frac{1}{10^3}}$$

$$= \frac{11205}{9990} = \frac{83}{74}$$

Non-Recurring Decimal

$$\sqrt{2} = 1.41421356\cdots$$

$$e = 2.718281828459045\cdots$$

The square root of a prime number is irrational (1)

Let p be a prime number.

Assume that $\sqrt{p}$ is rational.

(*) $\sqrt{p} = \dfrac{n}{m}$

Here the right−hand side is irreducible.

The square root of a prime number is irrational (2)

$(*) \;\Longrightarrow$

$(**) \quad n^2 = pm^2$

p **is a prime number**

n^2 **is a multiple of** $p \Leftrightarrow$

n **is a multiple of** p

$n = pa$ **+** $(**) \;\Longrightarrow$

$pm^2 = n^2 = p^2 a^2 \Longrightarrow$

$m^2 = pa^2$

The square root of a prime number is irrational (3)

$$m^2 = pa^2$$

implies that

m is a multiple of p:

$$m = pb$$

$$\Longrightarrow$$

$$\sqrt{p} = \frac{n}{m} = \frac{pa}{pb} = \frac{a}{b}$$

(contradiction)

Theory
of
Real Numbers

Main Theme

How do we characterize **irrational numbers** ?

What is the **convergence** of sequences ?

Completeness

Convergence of Sequences

Definition of Convergence

$\{a_n\}$ **sequence of real numbers**

$\{a_n\}$ **converges to** a

$$\overset{\text{def}}{\Longleftrightarrow}$$

$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$$\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$$

Cauchy’s Test

Cauchy's Test

$$\{a_n\} \text{ converges}$$

$$\Leftrightarrow$$

$$\lim_{n,m\to\infty} |a_n - a_m| = 0$$

Sequences

Sequences versus Functions

	Domain of Definition	Range
Sequence	Natural Numbers	Real Numbers
Functions	Real Numbers	Real Numbers

Definition

The sequnce $\{a_n\}$ **converges to** a

$$\overset{\text{def}}{\Longleftrightarrow}$$

$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$

Notation : $\lim\limits_{n \to \infty} a_n = a$

Fundamental Example

$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0}$$

Examples (1)

$$(1)\ \lim_{n\to\infty}\frac{n+1}{n}=1$$

$$(2)\ \lim_{n\to\infty}\frac{n+1}{n^2}=0$$

$$(3)\ \lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=0$$

Example (2)

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \textbf{if} \quad 0 < a < 1 \\ 1 & \textbf{if} \quad a = 1 \\ +\infty & \textbf{if} \quad a > 1 \end{cases}$$

Examples (3)

$$(1)\ \lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad \textbf{for} \quad a > 0$$

$$(2)\ \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Bounded Monotone Sequence

Fundamental Theorem

Every bounded, monotone increasing sequence itself converges.

$a_n \le \exists M$ （**Bounded**）

$a_n \le a_{n+1}$ （**Monotone increasing**）

Example (Napier's Number)

$$\boxed{a_n = \left(1 + \frac{1}{n}\right)^n}$$

$$\boxed{e = {}^{\exists}\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n}$$

Bounded Sequences

Fact

A convergent sequence is bounded.

Bolzano-Weierstrass Theorem

Bolzano（1781—1848）

Weierstrass（1815—1897）

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

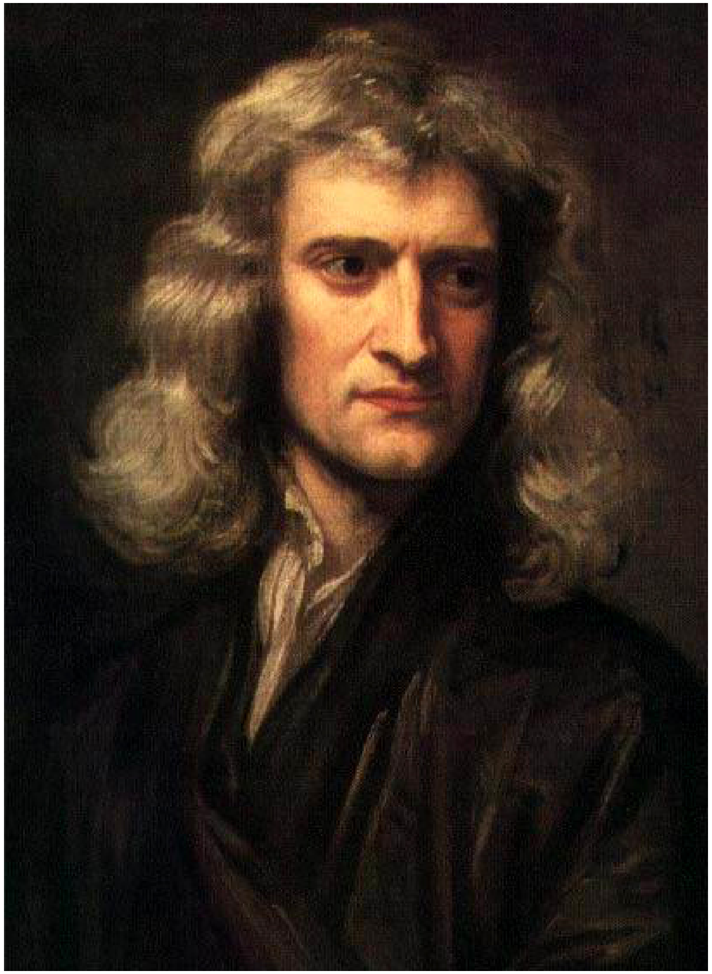
Numerical Analysis

Newton’s Method versus Bisection Method

Method	Newton’s Method	Bisection Method
Hypotheses	Differentiability Monotonicity	Continuity
Merits Demerits	Strong Hypotheses Rapid Convergence	Weak Hypotheses Slow Convergence
Background	Convergence of Monotone Sequences	Intermediate Value Theorem

Newton’s Approximation Method

Isaac Newton (1642-1727)



Newton's Approximation Method

$$r > 0,\ a_0 > 0$$

$$a_{n+1} := \frac{1}{2}\left(a_n + \frac{r}{a_n} \right), \quad n = 0, 1, 2, \ldots$$

$$\Longrightarrow$$

$$a_n \downarrow \sqrt{r} \quad (n \to \infty)$$

Example (Square root of 2)

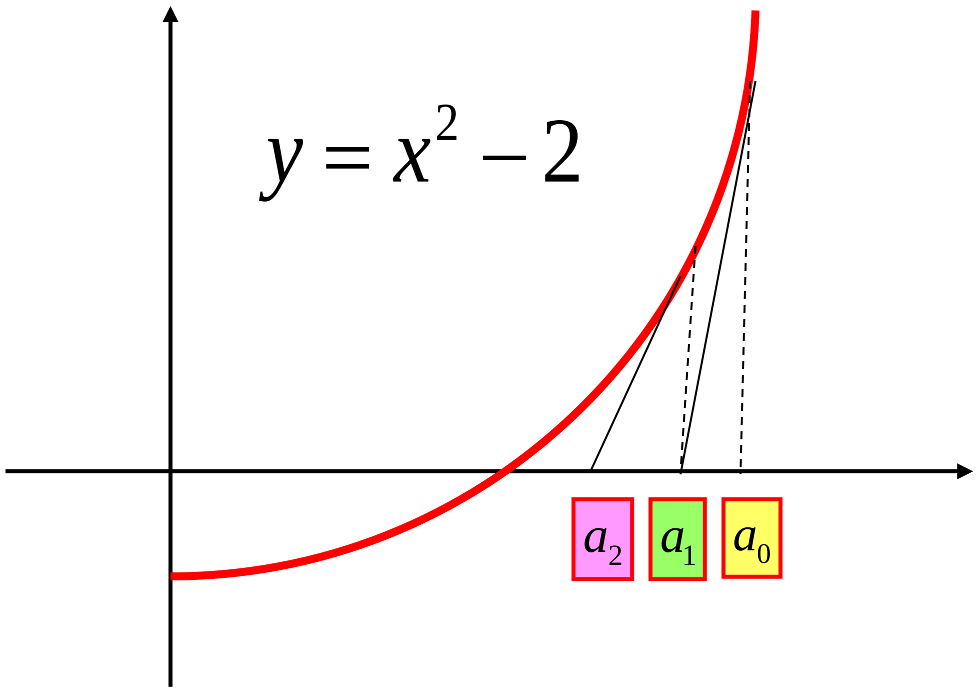
$$a_0 = 2, \quad a_1 = \frac{3}{2}$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n} \right)$$

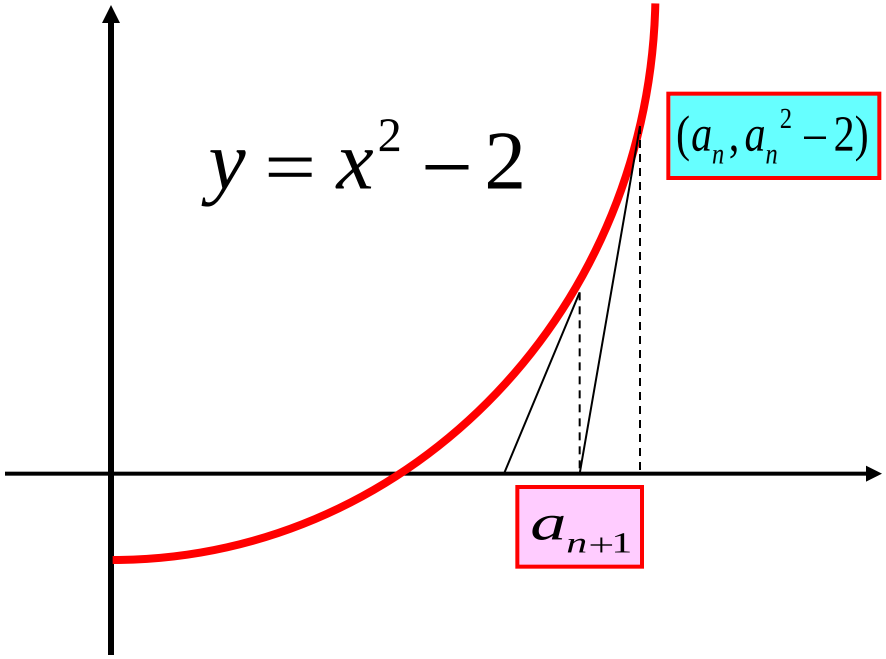
$$\Longrightarrow$$

$$\lim_{n\to\infty} = \sqrt{2}$$

Newton's Method (1)



Newton's Method (2)



Tangent Line at $(a_n, a_n^2 - 2)$:

$$y = 2a_n(x - a_n) + a_n^2 - 2 = 2a_nx - a_n^2 - 2$$

Bisection Method

Principle of Successive Subdivision

Cantor (1845—1918)

Cantor's Nested-Interval Property

$\{I_n\}$ **Sequence of closed intervals**

(1) $I_{n+1} \subset I_n$

(2) $|I_n| \to 0$

$\Longrightarrow$

$$\bigcap_{n=1}^{\infty} I_n = \{\textbf{One Point}\}$$

Sequence Version

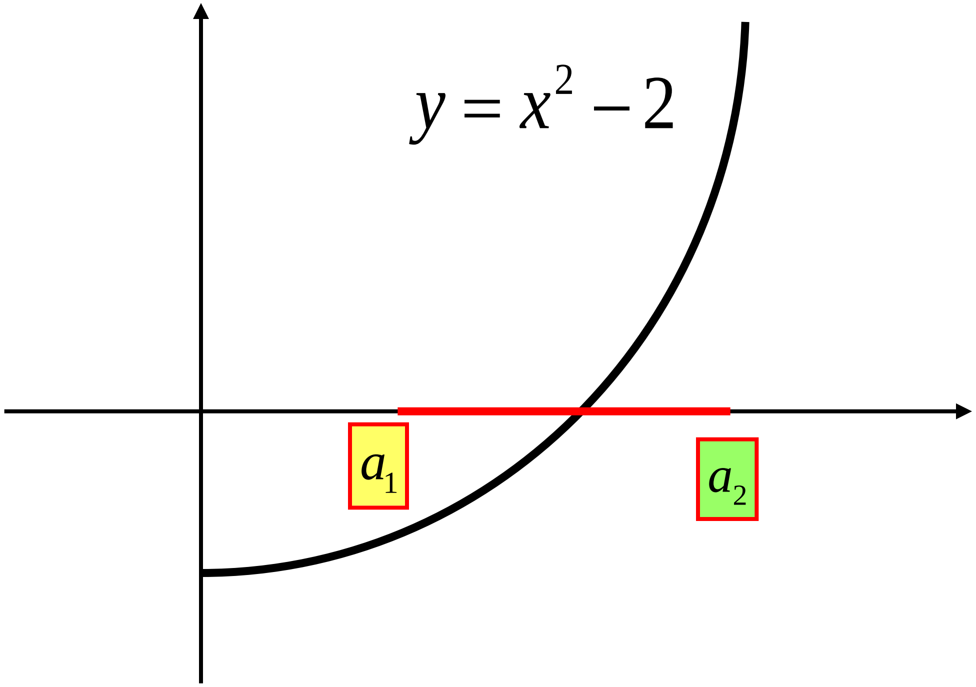
$$(1)\quad a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots \le b_{n+1} \le b_n \le b_2 \le b_1$$

$$(2)\quad b_n - a_n \to 0 \ \text{ as } n \to \infty$$

$$\Rightarrow$$

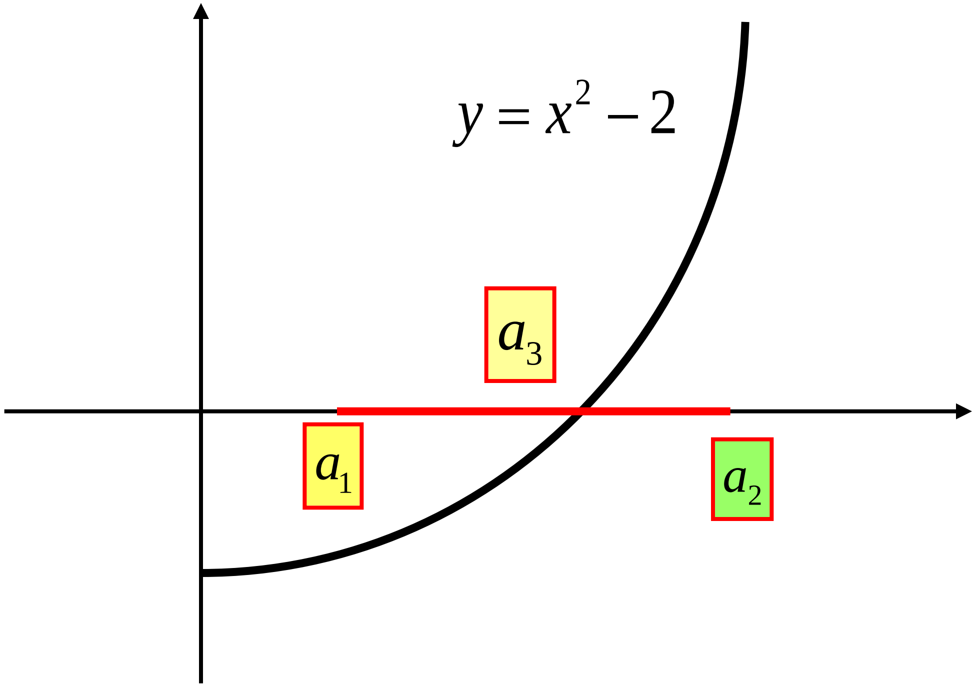
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Bisection Method（1）

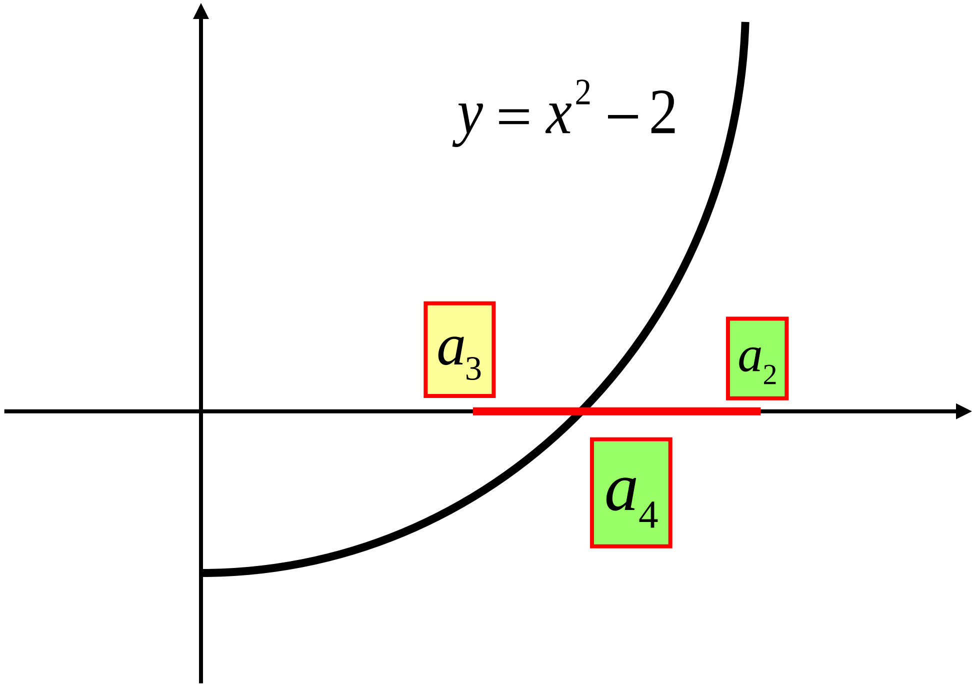


$\sqrt{2}$: **Square Root of 2**

Bisection Method（2）



Bisection Method (3)



Square Root of 2（1）

$$(1)\ \ 1^2 < 2 < 2^2 \Rightarrow 1 < \sqrt{2} < 2$$

$$\sqrt{2} \in I_1 = [1, 2]$$

$$(2)\ (1.4)^2 = 1.96 < 2 < (1.5)^2 = 2.25$$

$$\Rightarrow 1.4 < \sqrt{2} < 1.5$$

$$\sqrt{2} \in I_2 = [1.4, 1.5]$$

$$(3)\ (1.41)^2 = 1.9881 < 2 < (1.42)^2 = 2.0164$$

$$\Rightarrow 1.41 < \sqrt{2} < 1.42$$

$$\sqrt{2} \in I_3 = [1.41, 1.42]$$

Square Root of 2（2）

$$(n)\quad {a_n}^2 < 2 < {b_n}^2 \Longrightarrow a_n < \sqrt{2} < b_n$$

$$b_n - a_n = \frac{1}{10^n}$$

$$\sqrt{2} \in I_n = [a_n, b_n]$$

$$\Longrightarrow$$

$$\begin{cases} a_n \uparrow \alpha \\ b_n \downarrow \alpha \end{cases}$$

$$\alpha = \sqrt{2}$$

Complex Numbers

Carl Friedrich Gauss

Gauss

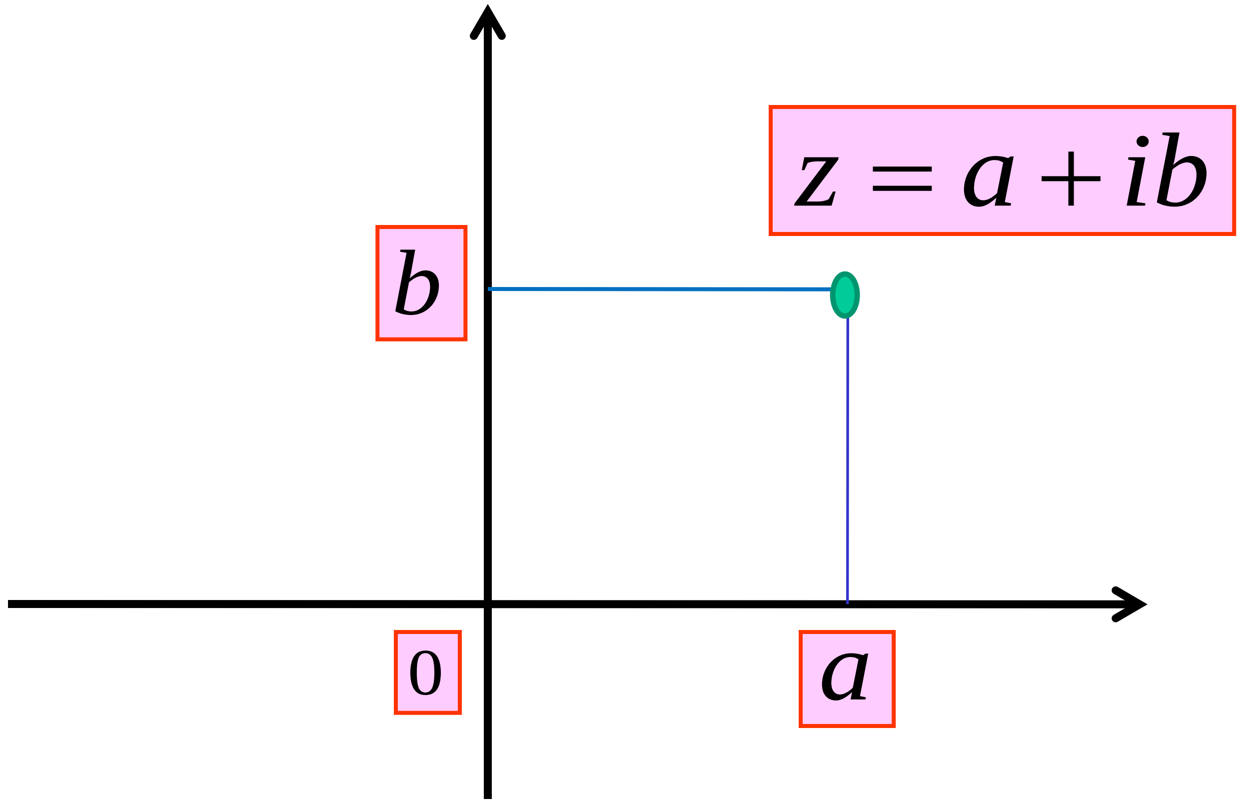
◆Carl Friedrich Gauss (1777-1855)
German Mathematician and Physicist

Complex Number

$$\boxed{\begin{aligned}&a + ib = c + id\\&\Leftrightarrow\\&a = c,\, b = d\end{aligned}}$$

$$\boxed{i = \sqrt{-1}}$$

Complex Plane

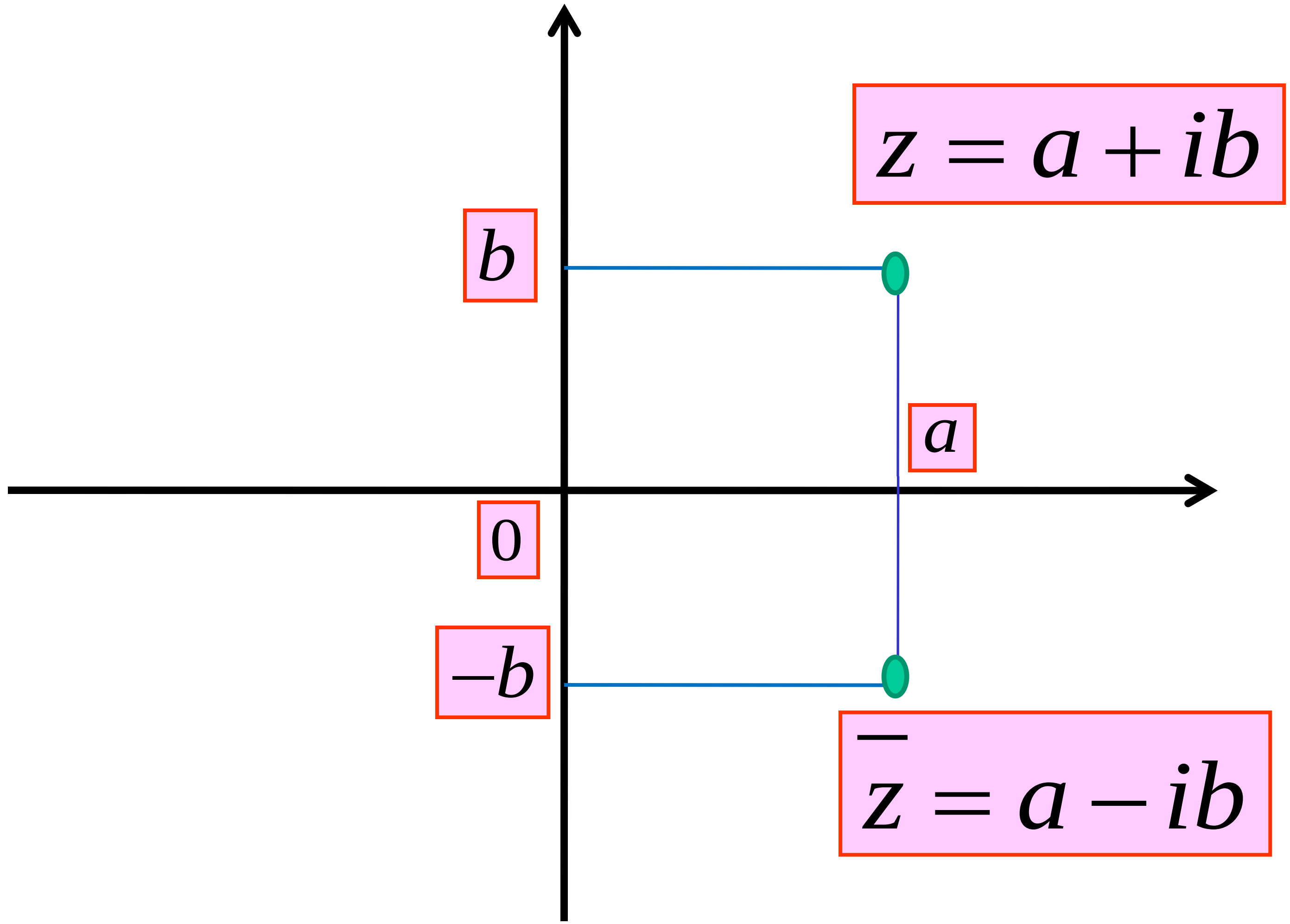


Conjugate of a Complex Number

$$z = a + ib$$

$$\Rightarrow$$

$$\bar{z} = a + i(-b) = a - ib$$

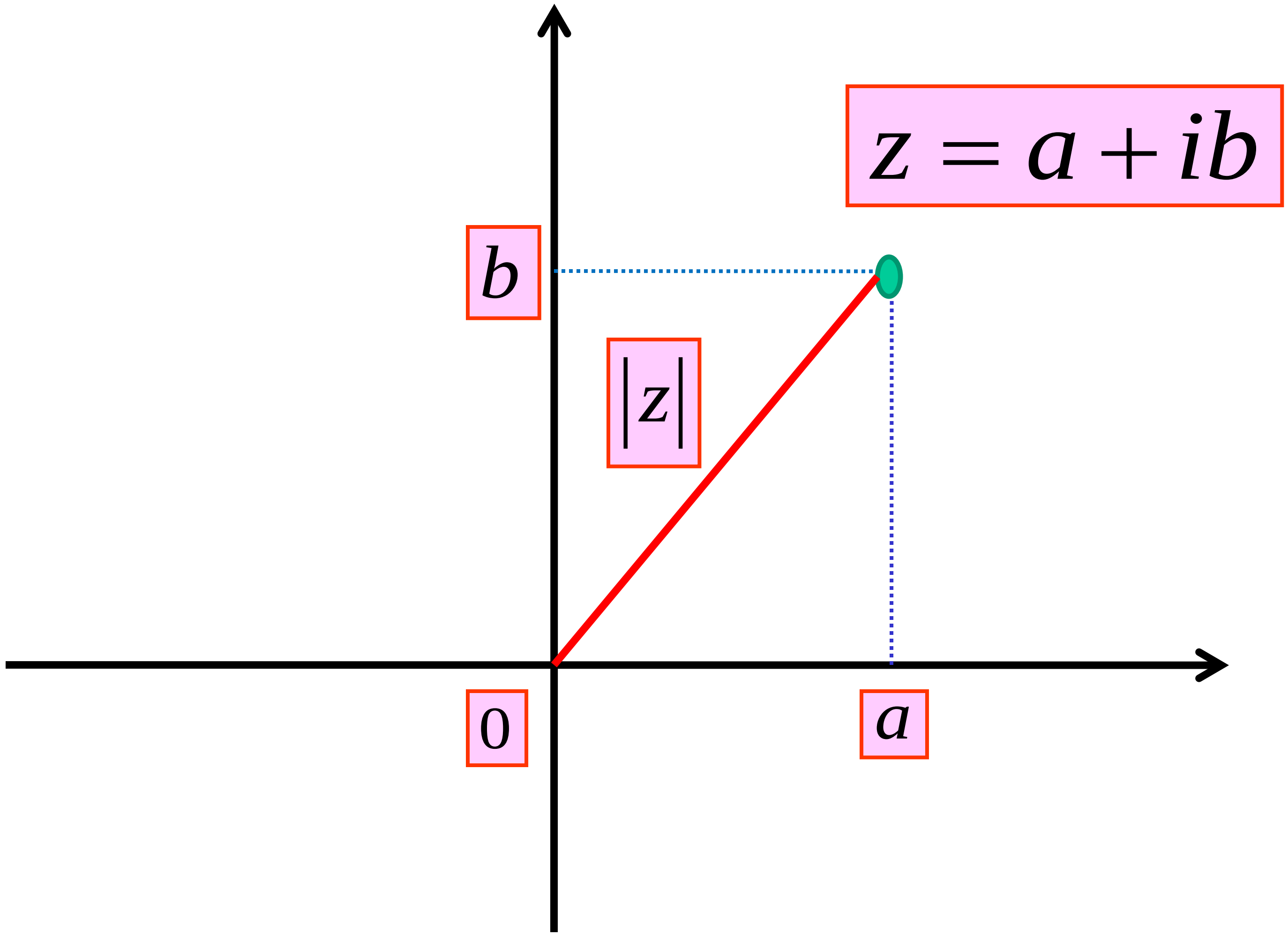

$z = a + ib$
b
a
0
$-b$
$\bar{z} = a - ib$

Absolute Value of a Complex Number

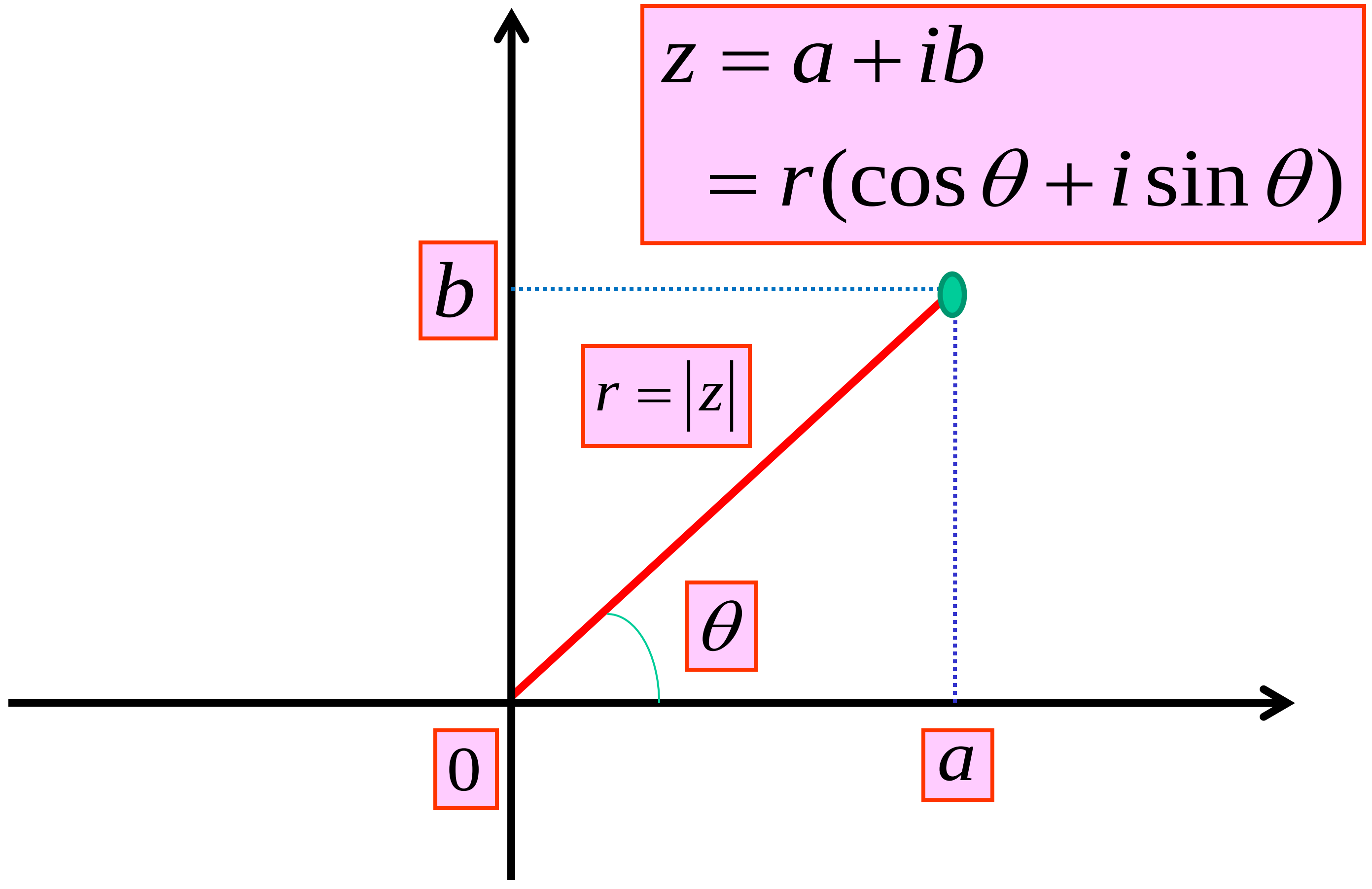
$$z = a + ib$$

$$\Rightarrow$$

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$


$z = a + ib$
b
$|z|$
0
a

Polar Coordinates of a Complex Number

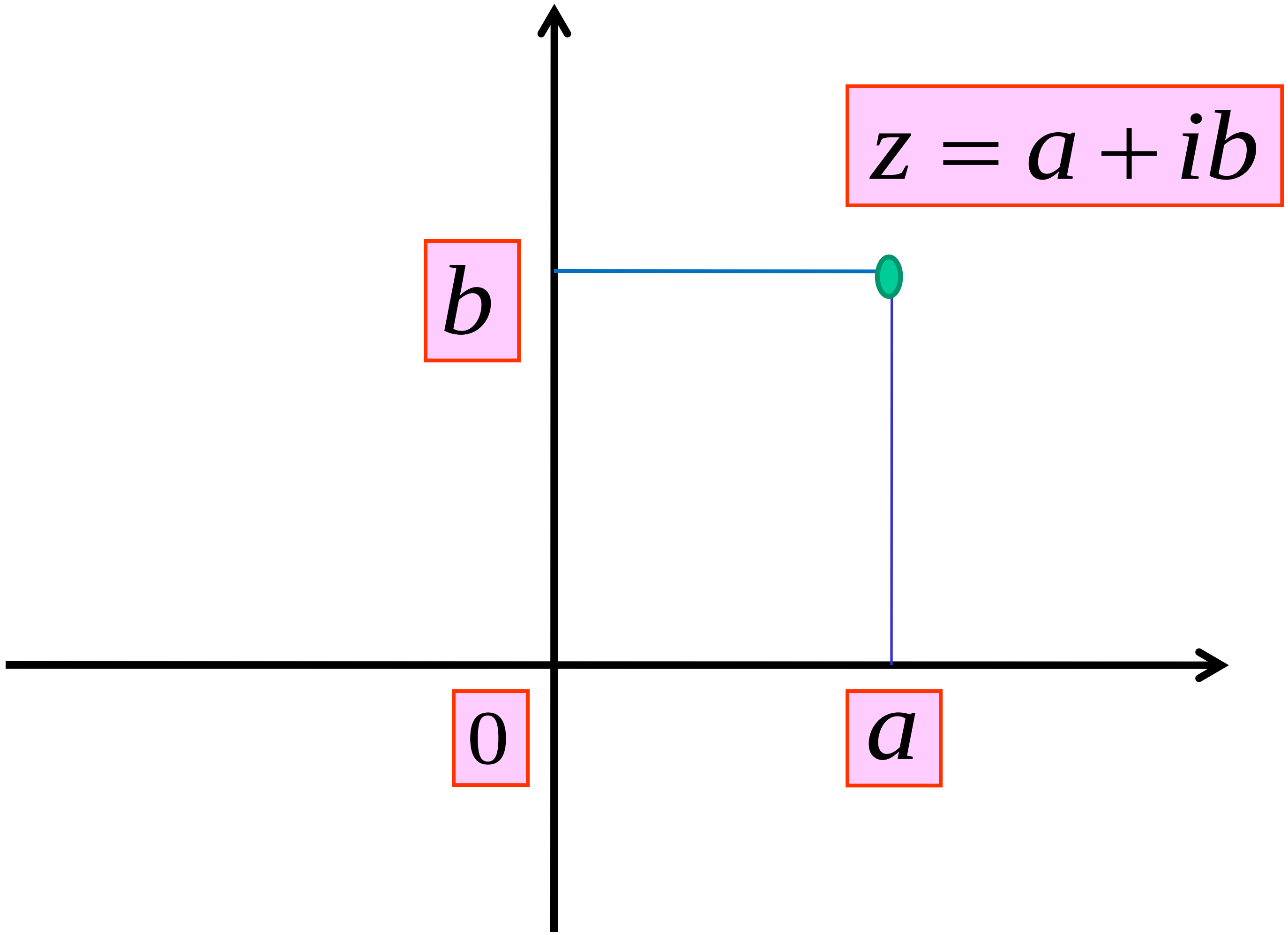


Sum of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

$$z + w = (a + c) + i(b + d)$$


$z = a + ib$
b
0
a

Difference of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

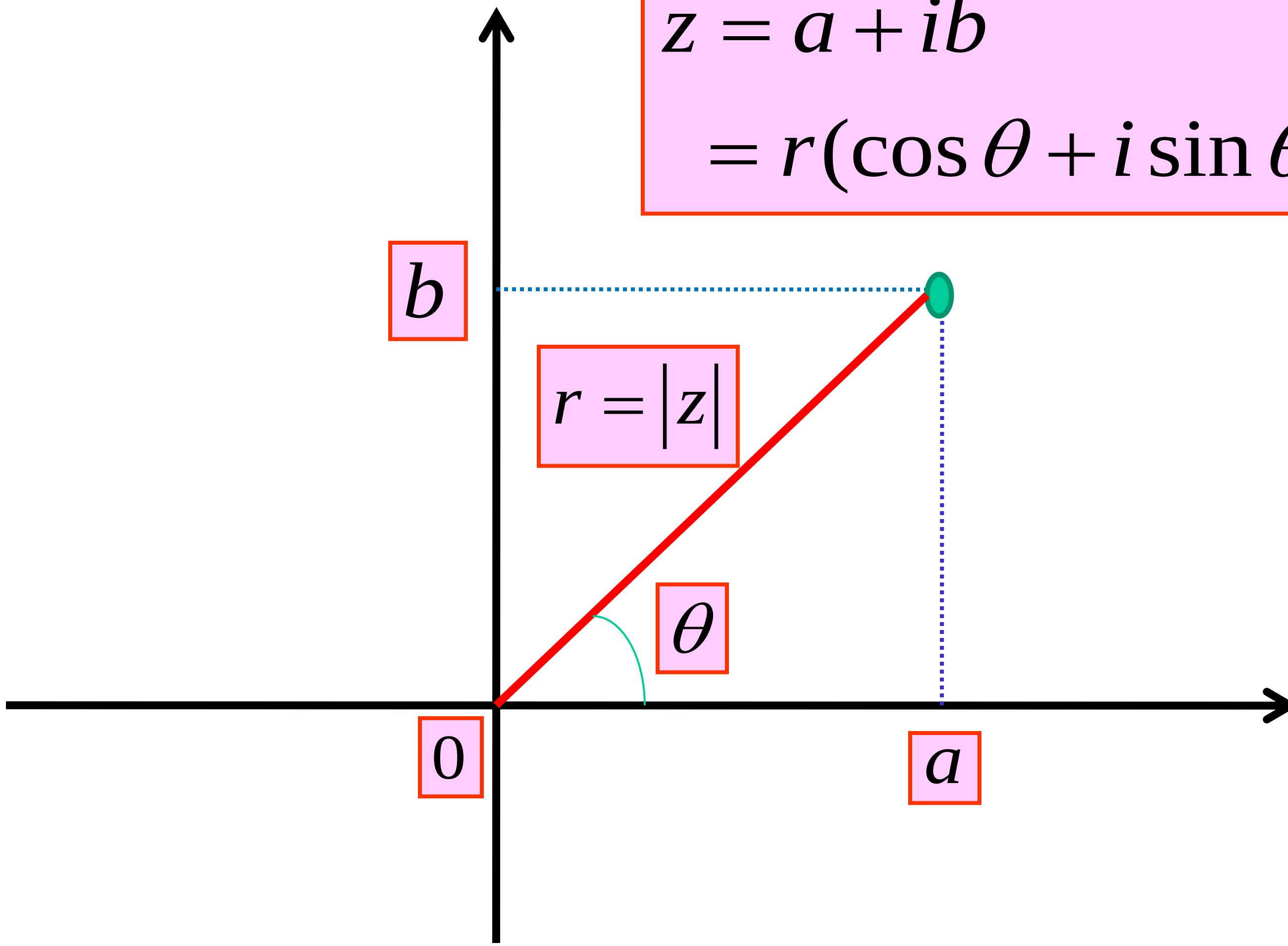
$$z - w = (a - c) + i(b - d)$$

Product of Complex Numbers

$$\boxed{\begin{aligned} & z = a + ib, \quad w = c + id \\ & \Rightarrow \\ & zw = (ac - bd) + i(ad + bc) \end{aligned}}$$

$$\boxed{i = \sqrt{-1} \Rightarrow i^2 = -1}$$

$$
\begin{aligned}
z &= a + ib \\
&= r(\cos\theta + i\sin\theta)
\end{aligned}
$$



Product of Complex Numbers

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$w = s(\cos\omega + i\sin\omega) = se^{i\omega}$$

$$\Longrightarrow$$

$$zw = rs(\cos(\theta+\omega) + i\sin(\theta+\omega))$$

$$= rse^{i(\theta+\omega)}$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\forall n \in \mathbf{Z}$$

Leonhard Euler (1707-1783)

Euler's Formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\boxed{e^{i\pi} = \cos\pi + i\sin\pi = -1}$$

Euler + De Moivre

$$\boxed{\begin{aligned}(e^{i\theta})^n &= (\cos\theta + i\sin\theta)^n \\ &= \cos n\theta + i\sin n\theta \\ &= e^{in\theta} \qquad (\forall n \in \mathbf{Z})\end{aligned}}$$

Algebraic Equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

$$a_i \in \mathbf{C}$$

Fundamental Theorem of Algebra (Gauss)

Every algebraic equation

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0, \; a_0 \neq 0$$

has n roots in C counted with multiplicity.

Example (1)

$$ax + b = 0,\ a \neq 0$$

$$\Longrightarrow$$

$$x = -\frac{b}{a}$$

Example (2)

$$ax^2 + bx + c = 0, \quad a \neq 0$$

$$\Rightarrow$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Imaginary Number

$$x^2 + 1 = 0$$

$$\Longrightarrow$$

$$x = \pm\sqrt{-1}$$

Canonical Forms of Polynomials of second-order

Mean Value Theorem

Taylor's Theorem

$\Downarrow$

Polynomial Approximation

Polynomial

$$\boxed{\begin{aligned} z &= f(x, y) \\ &= ax^2 + 2bxy + cy^2 \end{aligned}}$$

Matrix Form

$$z = f(x, y)$$

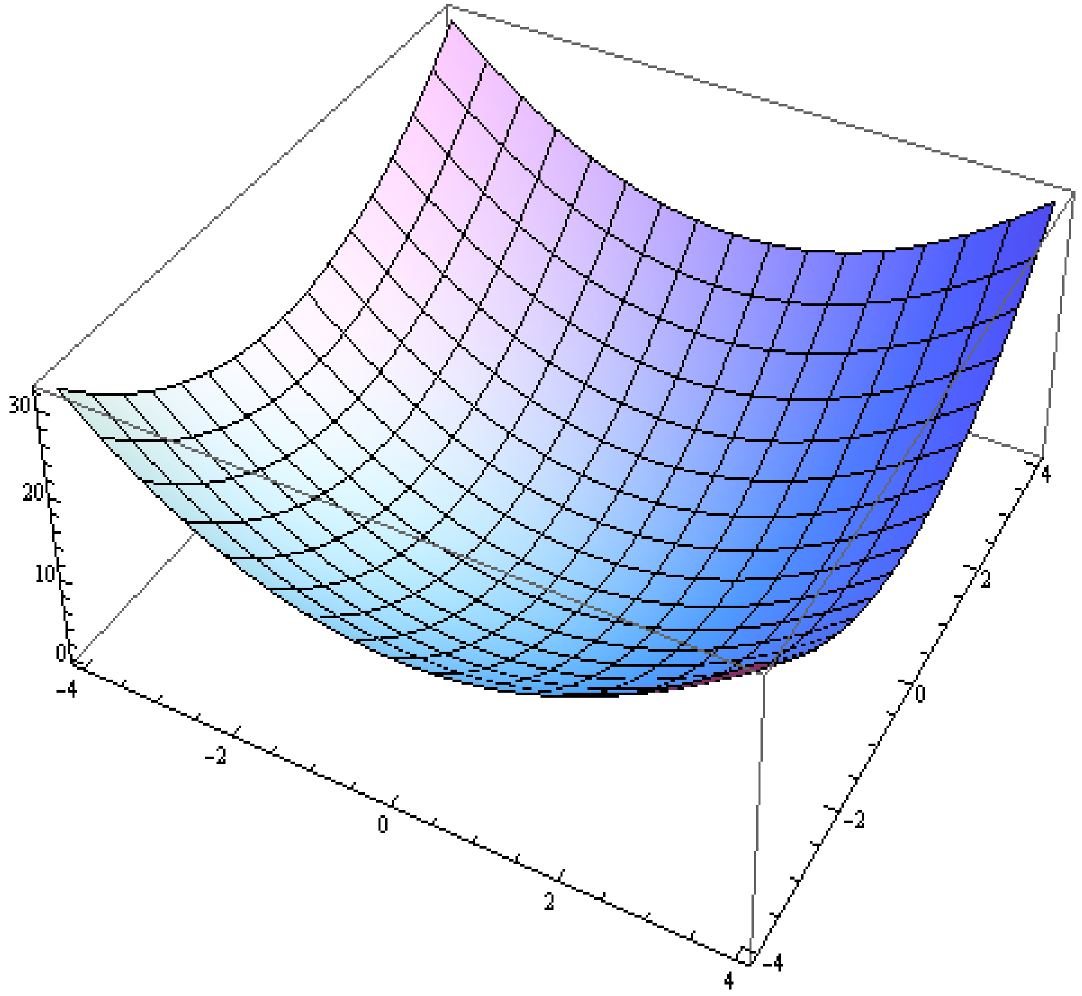
$$= ax^2 + 2bxy + cy^2$$

$$\Longrightarrow$$

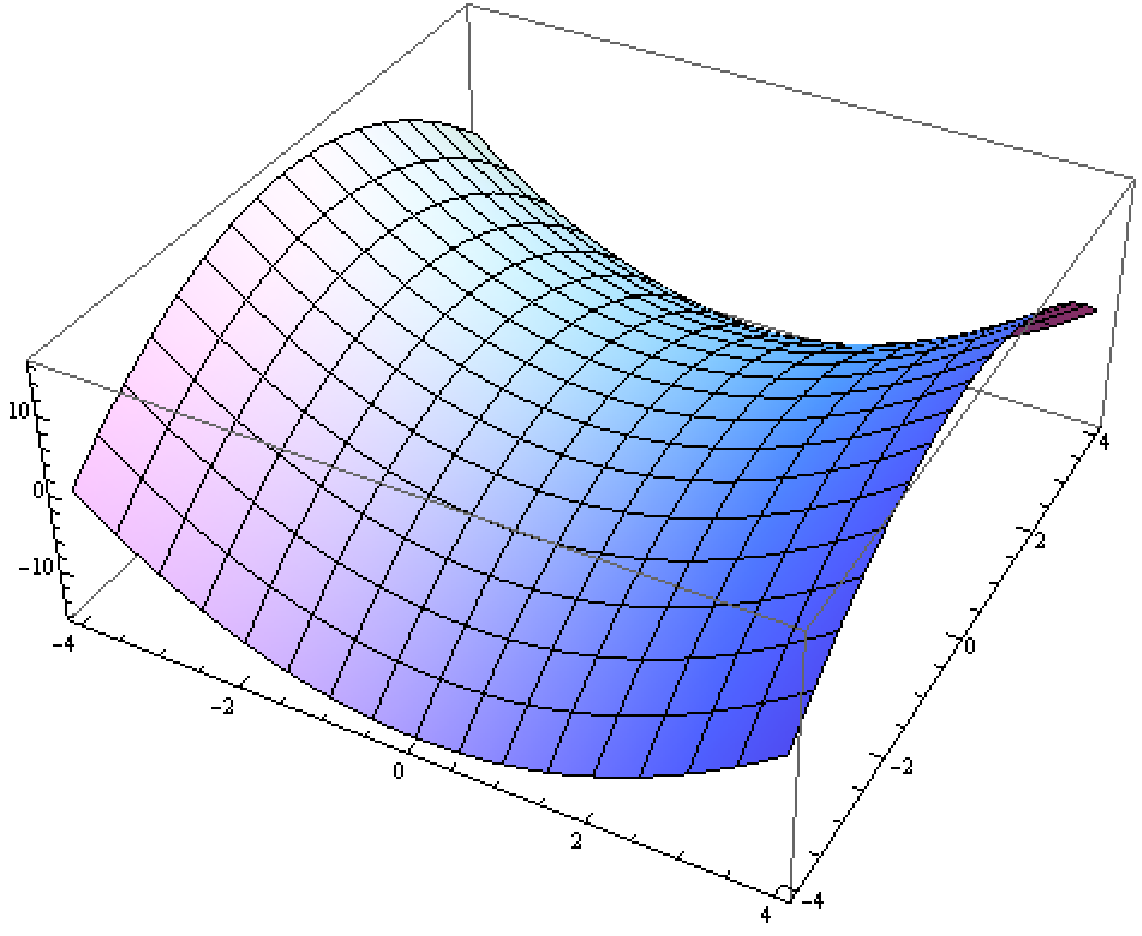
$$ax^2 + 2bxy + cy^2$$

$$= \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

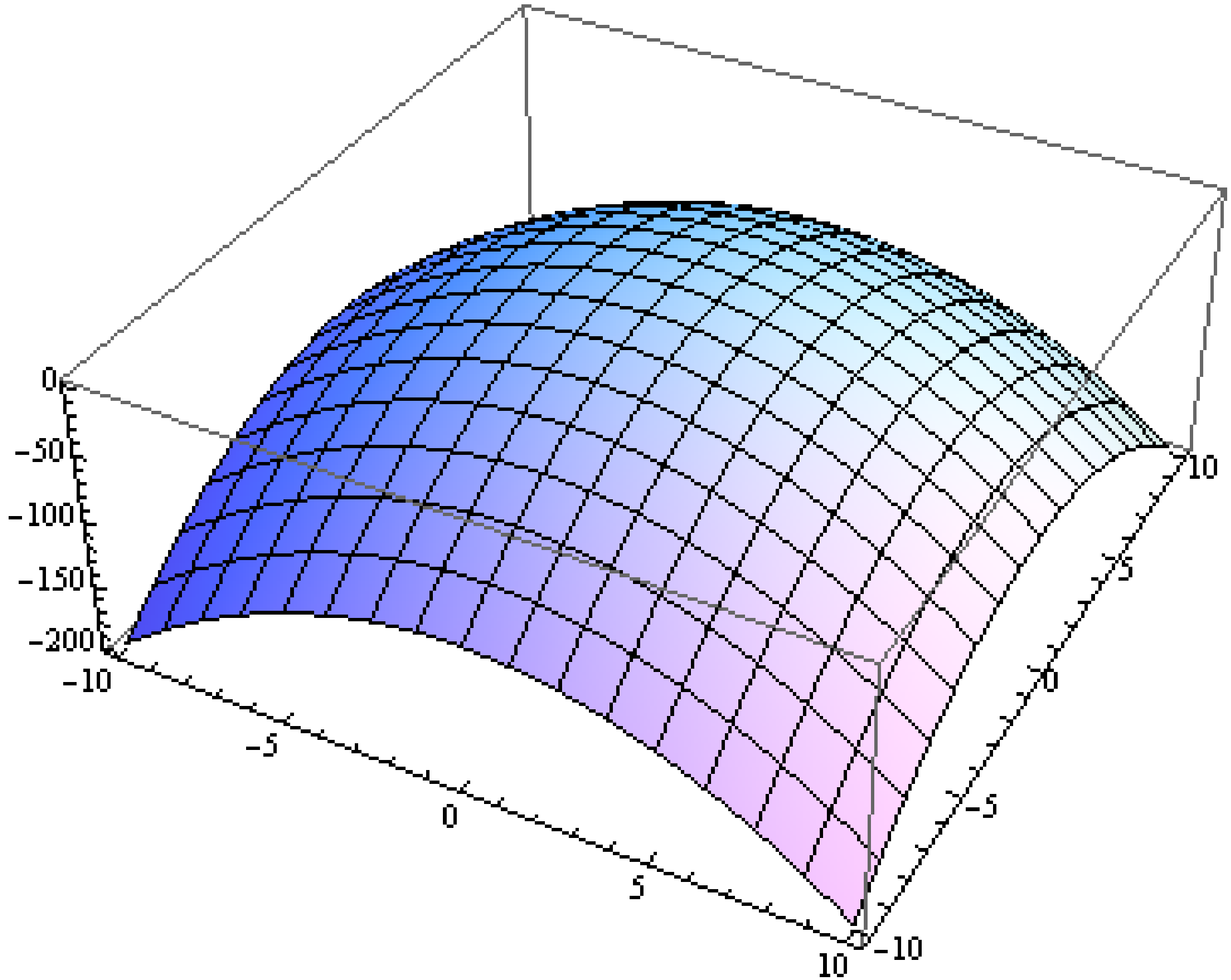
$z = x^2 + y^2$ **(minimal point)**



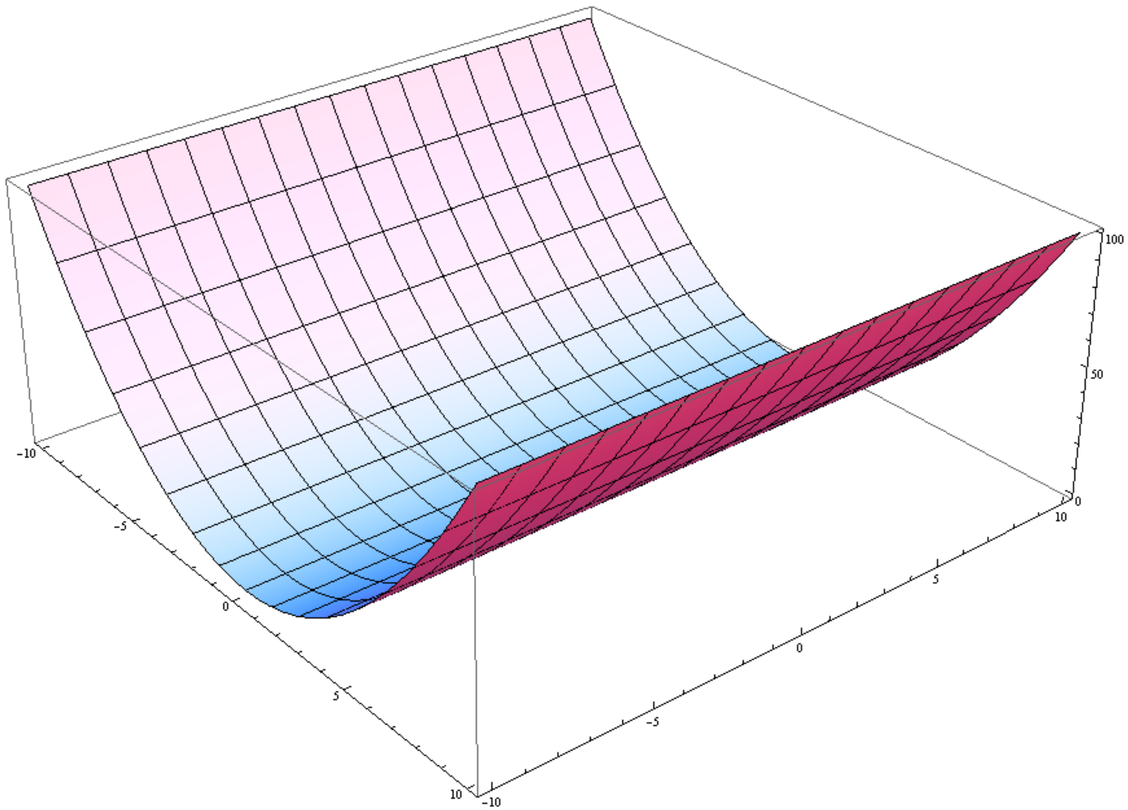
$z = x^2 - y^2$ （**saddle point**）



$$z = -x^2 - y^2 \quad \textbf{(maximal point)}$$



$z = x^2$ （degenerate point）



Theory of Matrices

General Form of a Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix}$$

Row of a Matrix

$$\boxed{\begin{pmatrix} a_{i1} & a_{i2} & \cdot & \cdot & a_{im} \end{pmatrix}}$$

Colum of a Matrix

$$\begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ a_{nk} \end{pmatrix}$$

Operations of Matrices

Sum of Matrices

$$A + B = \left(a_{ij} + b_{ij} \right)$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdot & \cdot & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdot & \cdot & a_{2m} + b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdot & \cdot & a_{nm} + b_{nm} \end{pmatrix}$$

Difference of Matrices

$$A - B = \left(a_{ij} - b_{ij} \right)$$

$$= \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdot & \cdot & a_{1m} - b_{1m} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdot & \cdot & a_{2m} - b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} - b_{n1} & a_{n2} - b_{n2} & \cdot & \cdot & a_{nm} - b_{nm} \end{pmatrix}$$

Scalar Multiple of a Matrix

$$\alpha A = \left(\alpha a_{ij}\right)$$

$$= \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdot & \cdot & \alpha a_{1m} \\ \alpha a_{21} & \alpha a_{22} & \cdot & \cdot & \alpha a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha a_{n1} & \alpha a_{n2} & \cdot & \cdot & \alpha a_{nm} \end{pmatrix}$$

Product of Matrices (1)

$$\mathbf{Ab}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdot\cdot + a_{1m}b_m \\ a_{21}b_1 + a_{22}b_2 + \cdot\cdot + a_{2m}b_m \\ \cdot \\ \cdot \\ a_{n1}b_1 + a_{n2}b_2 + \cdot\cdot + a_{nm}b_m \end{pmatrix}$$

Motivation

$$y_1 = 2x_1 + 5x_2$$

$$y_2 = -x_2$$

$$y_3 = -x_1 + 4x_2$$

$$\Longrightarrow$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 0 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Example

$$\mathbf{A}\mathbf{b} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Product of Matrices (2)

$$AB = \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right)$$

Zero Matrix

$$\boxed{O = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 \end{pmatrix}}$$

$$\boxed{A + O = O + A = A}$$

Unit Matrix

$$E_n = (\delta_{ij}) = \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 \end{pmatrix}$$

$$E_m A = A, \quad AE_n = A$$

Kronecker's Delta

$$\boxed{\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}}$$

$$\boxed{E = \left(\delta_{ij}\right)}$$

Inverse Matrix

$$A = \left(a_{ij}\right)_{1\le i,j\le n}, \quad B = \left(a_{ij}\right)_{1\le i,j\le n}$$

$$AB = BA = E_n$$

$$\Leftrightarrow$$

$B = A^{-1}$: inverse matrix of A

Uniqueness of an Inverse Matrix

$$AB_1 = B_1 A = E_n$$

$$AB_2 = B_2 A = E_n$$

$$\Longrightarrow$$

$$B_1 = B_1 E$$

$$= B_1 \left(AB_2 \right) = \left(B_1 A \right) B_2$$

$$= E_n B_2 = B_2$$

Transposed Matrix

$$A = (a_{ij})$$

$$\Longrightarrow$$

$${}^{t}A = (a_{ji})$$

Example

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 1 & 0 \\ -5 & 1 & 2 \end{pmatrix}$$

$$^{t}A = \begin{pmatrix} 3 & 4 & -5 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}$$

Symmetric Matrix

$$A = (a_{ij})$$

$$^{t}A = A$$

$$\Leftrightarrow$$

$$a_{ij} = a_{ji}$$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 3 \end{pmatrix}$$

Alternating Matrix

$$A = (a_{ij})$$

$${}^t A = -A$$

$$\Leftrightarrow$$

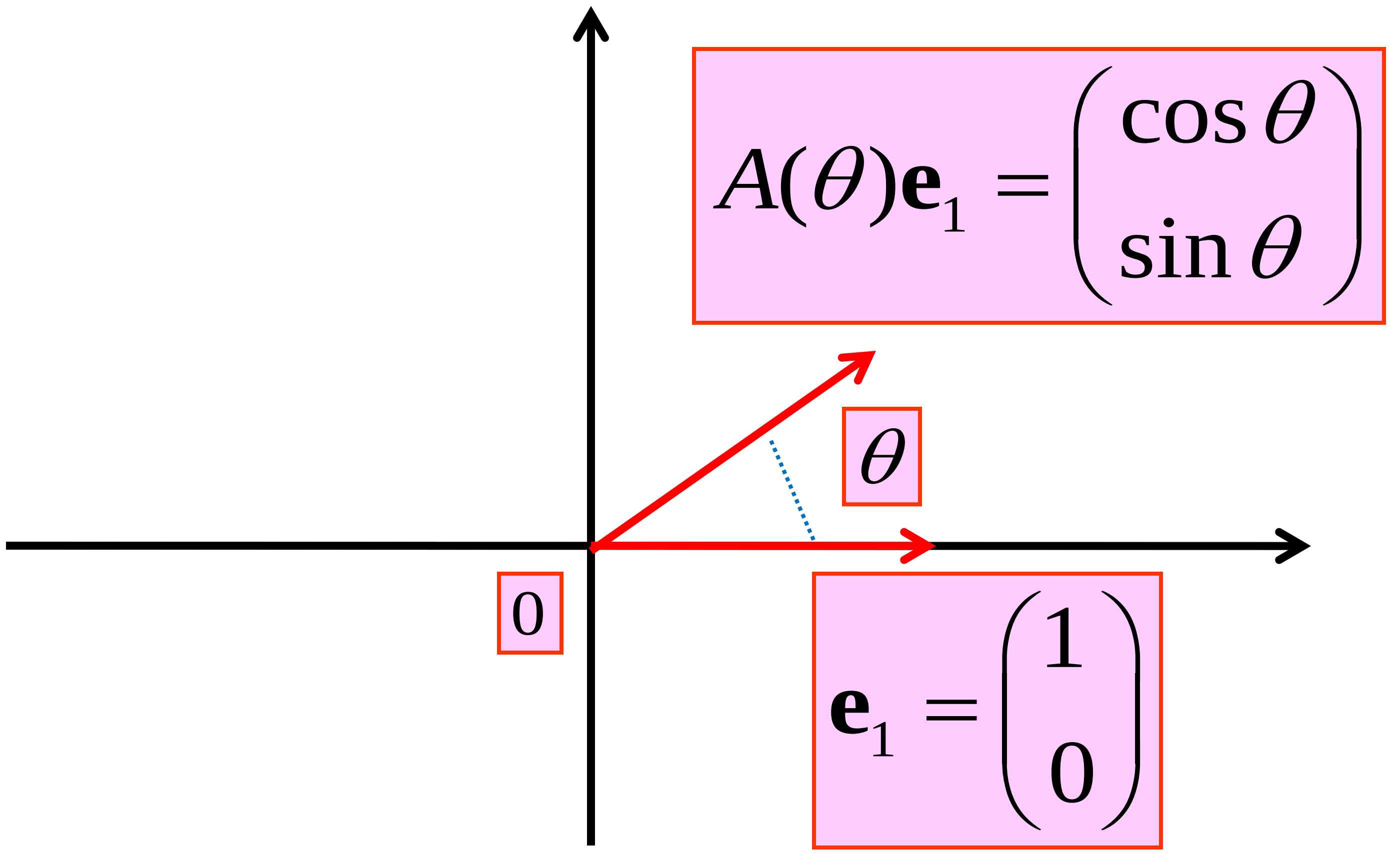
$$a_{ij} = -a_{ji}$$

Example

$$\begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix}$$

Addition Theorem of Trigonometric Functions

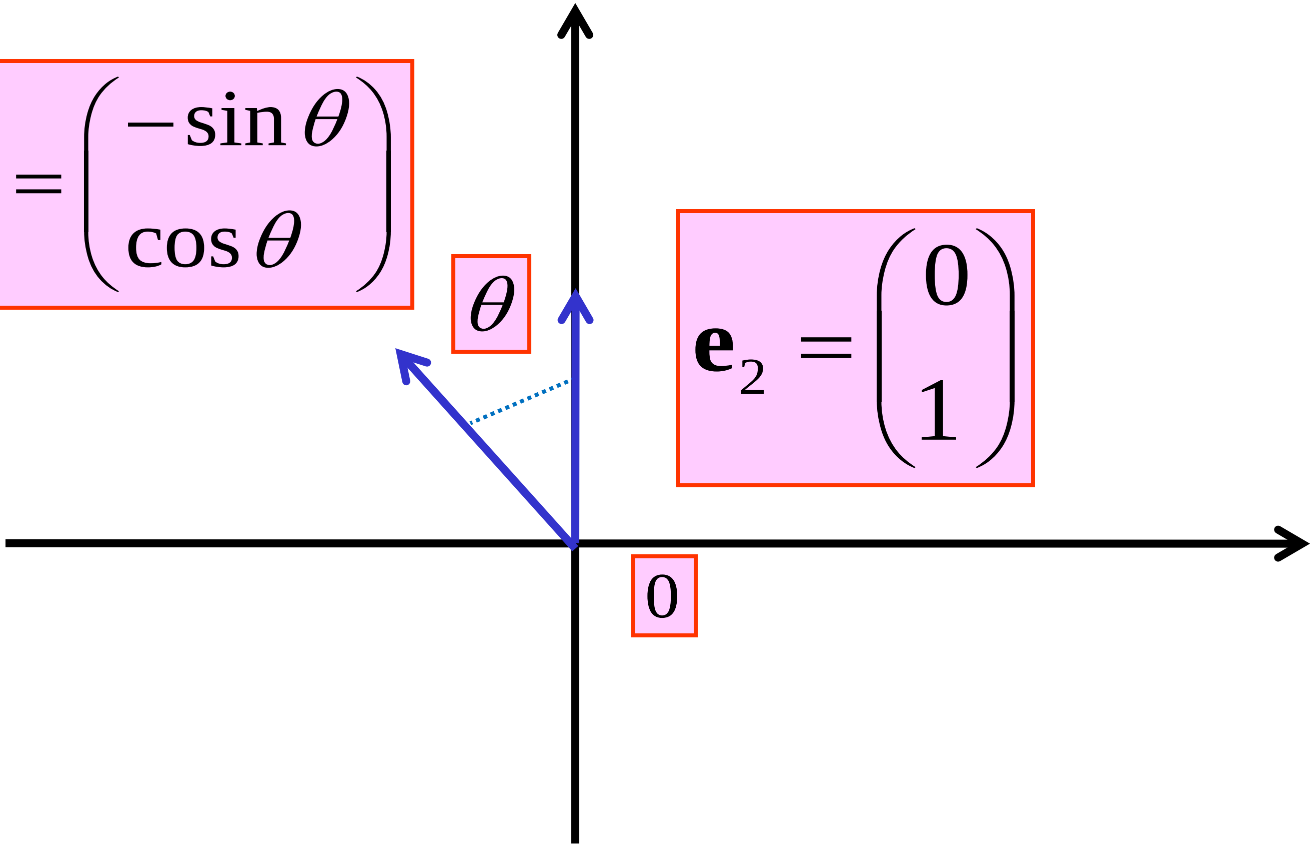
Rotation (1)



$A(\theta)\mathbf{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
θ
0
$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Rotation (2)

$$A(\theta)\mathbf{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Matrix of Rotation (1)

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotation of θ

Matrix of Rotation (2)

$$A(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A(\theta)^{-1}$$

Rotation of $-\theta$

Composition of Rotations (1)

$$A(\alpha)\mathbf{e}_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Composition of Rotations (2)

$$A(\beta)\left(A(\alpha)\mathbf{e}_1\right)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha \end{pmatrix}$$

Composition of Rotations (3)

$$A(\alpha)\mathbf{e}_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

Composition of Rotations (4)

$$A(\beta)\left(A(\alpha)\mathbf{e}_2\right)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{pmatrix}$$

Composition of Rotations (5)

$$A(\beta)A(\alpha)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{pmatrix}$$

Composition of Rotations (6)

$$\boxed{\begin{aligned} A(\beta)\big(A(\alpha)\mathbf{e}_1\big) &= A(\beta)A(\alpha)\mathbf{e}_1 \\ A(\beta)\big(A(\alpha)\mathbf{e}_2\big) &= A(\beta)A(\alpha)\mathbf{e}_2 \end{aligned}}$$

Composition of Rotations (7)

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$A(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$\Rightarrow$$

$$A(\alpha)A(\beta) = A(\beta)A(\alpha) = A(\alpha+\beta)$$

Composition of Rotations (8)

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= A(\alpha+\beta)$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

Addition Theorem (1)

$$\boxed{\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}}$$

Addition Theorem (2)

$$\boxed{\begin{aligned} \sin A + \sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \\ \sin A - \sin B &= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \end{aligned}}$$

Addition Theorem (3)

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Addition Theorem (4)

$$\sin A\ \sin B = \frac{1}{2}\big(\cos(A - B) - \cos(A + B)\big)$$

$$\cos A \cos B = \frac{1}{2}\big(\cos(A - B) + \cos(A + B)\big)$$

$$\sin A \cos B = \frac{1}{2}\big(\sin(A + B) + \sin(A - B)\big)$$

Addition Theorem (5)

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin 2A = 2\sin A \cos A$$

Addition Theorem (6)

$$\boxed{\begin{aligned}\tan(A+B) &= \frac{\tan A + \tan B}{1-\tan A \tan B} \\ 1+\tan^2 A &= \frac{1}{\cos^2 A}\end{aligned}}$$

Inverse Matrices

Algorithm for Inverse Matrices

Left Elementary Transformations

(1) Interchange two rows
(2) Multiply a row by a non-zero constant
(3) Add a row by a multiplied another row

Gauss' Method (1)

$$(A, E) \underset{\text{Left Elemntary Transformations}}{\Longrightarrow} (E, B)$$

Gauss' Method (2)

$$C(A, E) = (CA, C) = (E, B)$$

$$\Longrightarrow$$

$$\begin{cases} CA = E \\ C = B \end{cases}$$

$$\Longrightarrow$$

$$BA = E$$

$$\Longrightarrow$$

$$B = A^{-1}$$

Example of Left Elementary Transformations

(1) Interchange two rows

$$\boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

Interchange two rows

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

(2) Multiply a row by a non-zero constant

$$\boxed{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda \neq 0}$$

Multiply a row by a non-zero constant

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{pmatrix}$$

(3) Add a row by a multiplied another row

$$\boxed{\begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Add a row by a multiplied another row

$$\begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} a+\lambda d & b+\lambda e & c+\lambda f \\ d & e & f \\ g & h & i \end{pmatrix}$$

Examples

Example 1

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & 1 & 2 \end{pmatrix}$$

$$(A, E) = \begin{pmatrix} 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -5 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$(E, A^{-1})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & -3 & 3 \end{pmatrix}$$

Inverse Matrix

$$A^{-1} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & 3 \end{pmatrix}$$

Example 2

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

$$(A, E) = \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$(E, A^{-1})$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -3 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

Inverse Matrix

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & -3 & 2 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Computational Approach

Numerical Computing with BASIC

Example

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

1 行を 2 で割る

1	0	.5	0
0	−1	1	−2
1	0	1	0
0	1	−1	3

2 行から 1 行の 0 倍を引く
3 行から 1 行の 1 倍を引く
4 行から 1 行の 0 倍を引く

1	0	.5	0
0	−1	1	−2
0	0	.5	0
0	1	−1	3

2 行を−1 で割る

1	0	.5	0
0	1	−1	2
0	0	.5	0
0	1	−1	3

1 行から 2 行の 0 倍を引く
3 行から 2 行の 0 倍を引く
4 行から 2 行の 1 倍を引く

1	0	.5	0
0	1	−1	2
0	0	.5	0
0	0	0	1

3 行を .5 で割る

1	0	.5	0
0	1	−1	2
0	0	1	0
0	0	0	1

1 行から 3 行の .5 倍を引く
2 行から 3 行の−1 倍を引く
4 行から 3 行の 0 倍を引く

1	0	0	0
0	1	0	2
0	0	1	0
0	0	0	1

4 行を 1 で割る

1	0	0	0
0	1	0	2
0	0	1	0
0	0	0	1

1 行から 4 行の 0 倍を引く
2 行から 4 行の 2 倍を引く
3 行から 4 行の 0 倍を引く

B

1	0	−1	0
−1	−3	2	−2
−1	0	2	0
0	1	0	1

Inverse Matrix

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & -3 & 2 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

System of Linear Equations and Ranks

System of Linear Equations

$$\boxed{\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}}$$

Coefficient Matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \end{pmatrix}$$

Idea of Rank (1)

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

$$\Leftrightarrow$$

$$x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Idea of Rank (2)

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

$$\Leftrightarrow$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A}$$

Rank of Matrices

Definition of Rank

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix}$$

$$\Longrightarrow$$

Left Elemntary Transformations

Matrix after Left Elementary Transformations (Echelon Form)

$$\begin{pmatrix} 1 & 0 & \cdot & 0 & c_{1r+1} & \cdot\cdot & c_{1n} \\ 0 & 1 & \cdot & \cdot & c_{2r+1} & \cdot\cdot & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & \cdot & 1 & c_{rr+1} & \cdot\cdot & c_{rn} \\ 0 & 0 & \cdot & \cdot & 0 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \cdot\cdot & 0 \end{pmatrix}$$

$$\boxed{\text{rank}\,A = \textbf{Number of } 1}$$

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation ↓ ↑ **Original Form**

Placement of Lines and Planes

Example 1

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & -1 \\ 1 & 3 & 1 & 2 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\operatorname{rank} A = 3}$$

Example 2

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\operatorname{rank} A = 3}$$

Computational Approach

Numerical Computing
with
BASIC

Example 1

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

2 行と 1 行を入れ替える

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{matrix}$$

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 2 \end{matrix}$$

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

4 行を-1 倍し，3 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Rank A = 3

Matrix after Left Elementary Transformations

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$\boxed{\operatorname{rank} A = 3}$$

Example 2

$$B = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

2 行と 1 行を入れ替える

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{matrix}$$

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 2 & -3 \end{matrix}$$

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{matrix}$$

4 行を-1 倍し，3 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Rank B = 3

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\operatorname{rank} B = 3}$$

Example 3

$$C = \begin{pmatrix} 1 & -2 & -3 & 4 \\ 2 & 3 & 1 & 1 \\ 3 & -4 & -7 & 10 \end{pmatrix}$$

2 行を 1 倍し，1 行の 2 倍を引く

3 行を 1 倍し，1 行の 3 倍を引く

$$\begin{matrix} 1 & -2 & -3 & 4 \\ 0 & 7 & 7 & -7 \\ 0 & 2 & 2 & -2 \end{matrix}$$

3 行を 7 倍し，2 行の 2 倍を引く

$$\begin{matrix} 1 & -2 & -3 & 4 \\ 0 & 7 & 7 & -7 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Rank C = 2

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\operatorname{rank} C = 2}$$

Example 4

$$D = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$\boxed{\operatorname{rank} D = 3}$$

System of Linear Equations and Geometry

Idea of Linear Algebra

System of Linear Equations

Matrix Representation ↓

↑ **Original Form**

Placement of Lines

Classification of Intersections

$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2$	**One-Point**
$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$	**Parallel Two Lines**
$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 2$	**Superposed Two Lines**

$$\boxed{\operatorname{rank} A \le \operatorname{rank} \tilde{A} \le \operatorname{rank} A + 1}$$

Equation of a Line (1)

$$\boxed{ax + by = c}$$

Equation of a Line (2)

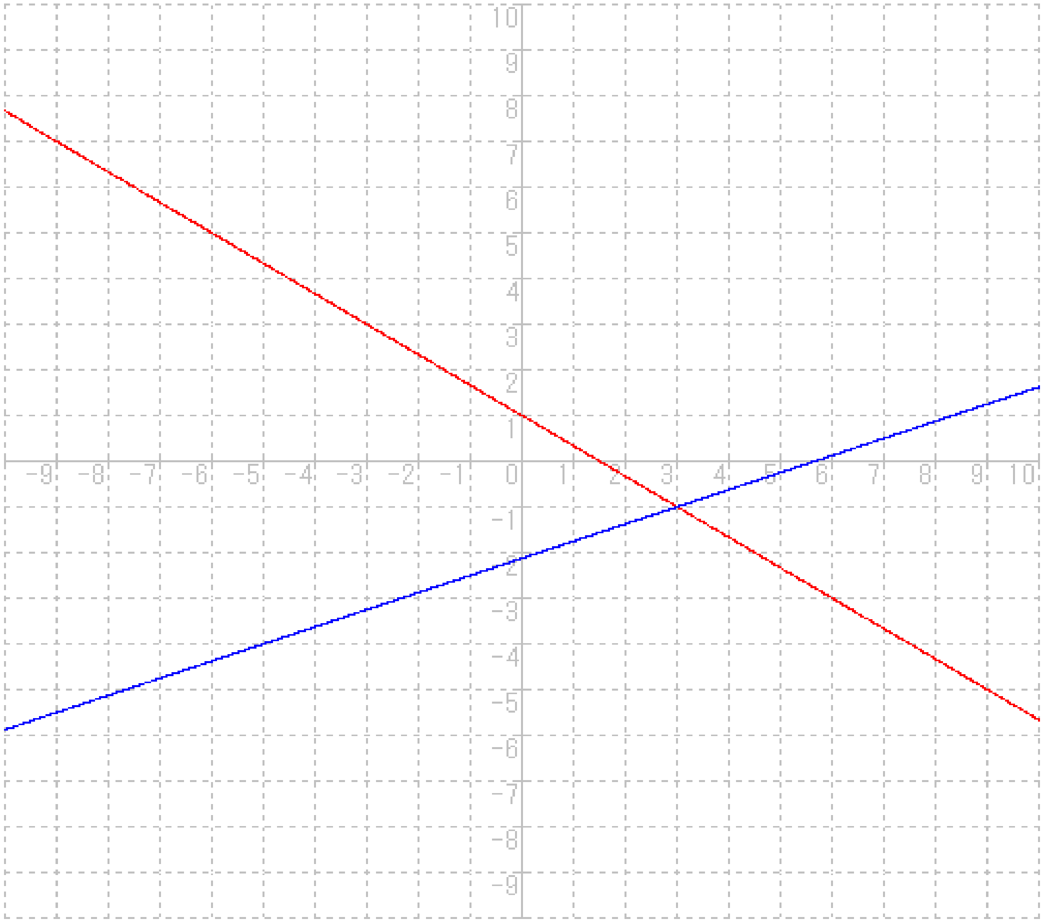
$$\begin{pmatrix} a \\ b \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = c$$

inner product

One-Point Intersection

$$\boxed{\begin{aligned} 2x + 3y &= 3 \\ 3x - 8y &= 17 \end{aligned}}$$

One-Point Intersection



$$\text{rank}\, A = \text{rank}\, \tilde{A} = 2$$

Coefficient Matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 3 & -8 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

Unique Solution

$$\tilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

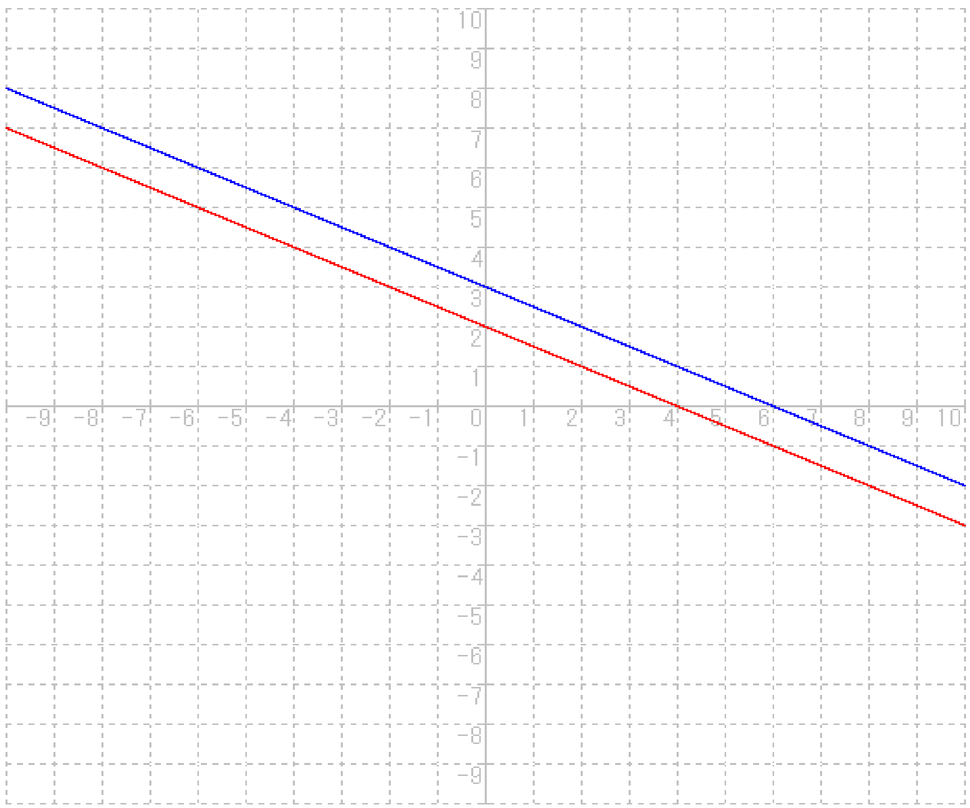
$$\boxed{\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2}$$

Parallel Two Lines

$$x + 2y = 2$$

$$x + 2y = 3$$

Parallel Two Lines



$$\boxed{\text{rank } A = 1 < \text{rank } \tilde{A} = 2}$$

Coefficient Matrix

$$\boldsymbol{A} = \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

No Solution

$$\widetilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ **(Impossible)**

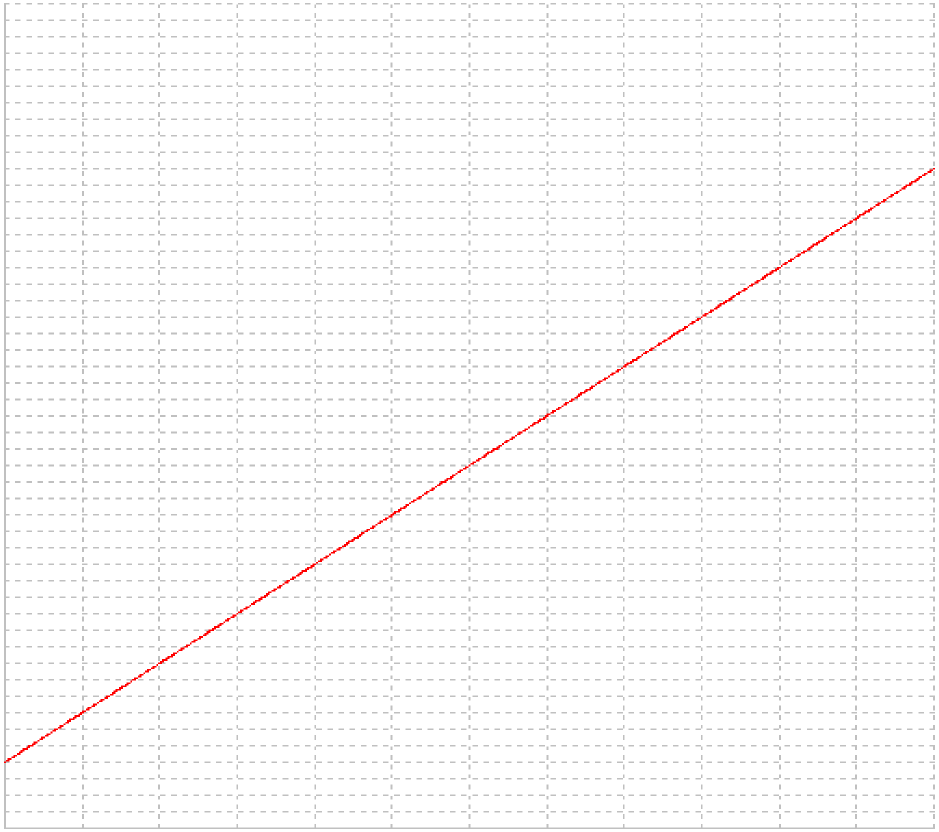
$$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$$

Superposed Two Lines

$$6x - 2y = -8$$

$$3x - y = -4$$

Superposed Two Lines



$$\text{rank}\,A = \text{rank}\,\tilde{A} = 1 < 2$$

Coefficient Matrix

$$\boldsymbol{A} = \begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

Many Solutions

$$\tilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & -1/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

(Indefinite)

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 2$$

System of Linear Equations

$$a_1 x + b_1 y + c_1 z = \alpha$$

$$a_2 x + b_2 y + c_2 z = \beta$$

$$a_3 x + b_3 y + c_3 z = \gamma$$

Equation of a Plane (1)

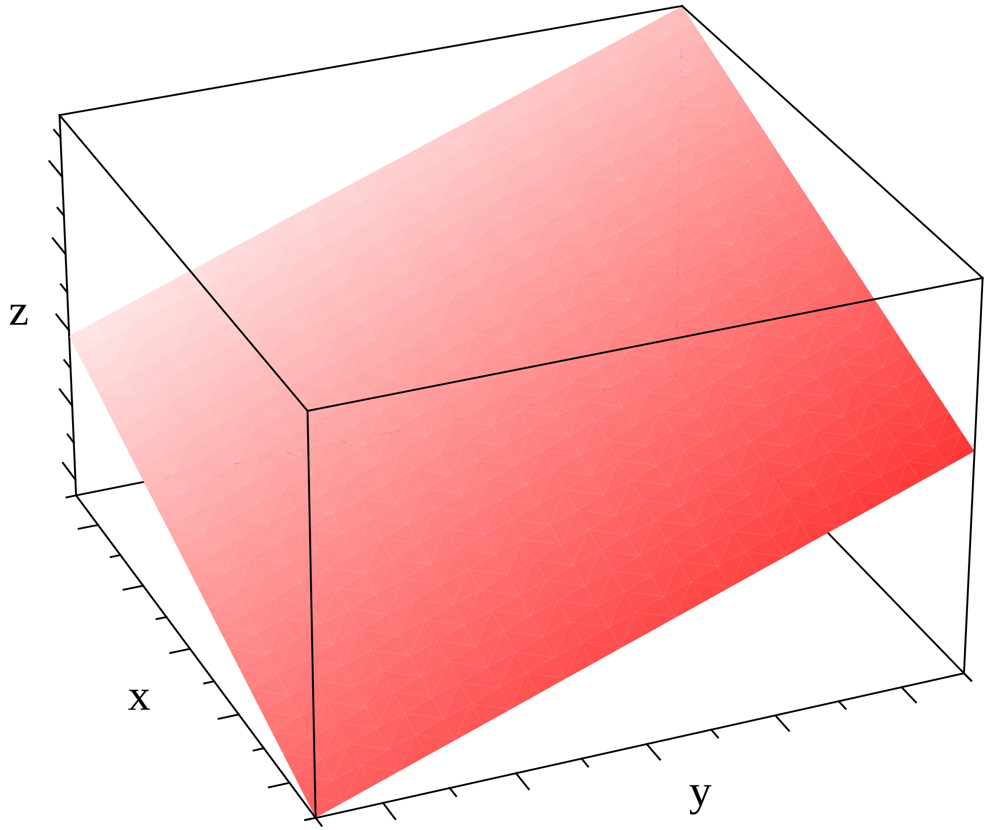
$$\boxed{ax + by + cz = d}$$

Equation of a Plane (2)

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d$$

inner product

Plane



Idea of Linear Algebra

System of Linear Equations

Matrix Representation ↓ ↑ **Original Form**

Placement of Planes

Classification of Intersections

$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 3$	**One-Point**
$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2 < 3$	**One Line**
$\operatorname{rank} A = 2 < \operatorname{rank} \tilde{A} = 3$	**Parallel Two Lines** **Parallel Three Lines**
$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 3$	**Superposed Three Planes**
$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$	**Parallel Two Planes** **Parallel Three Planes**

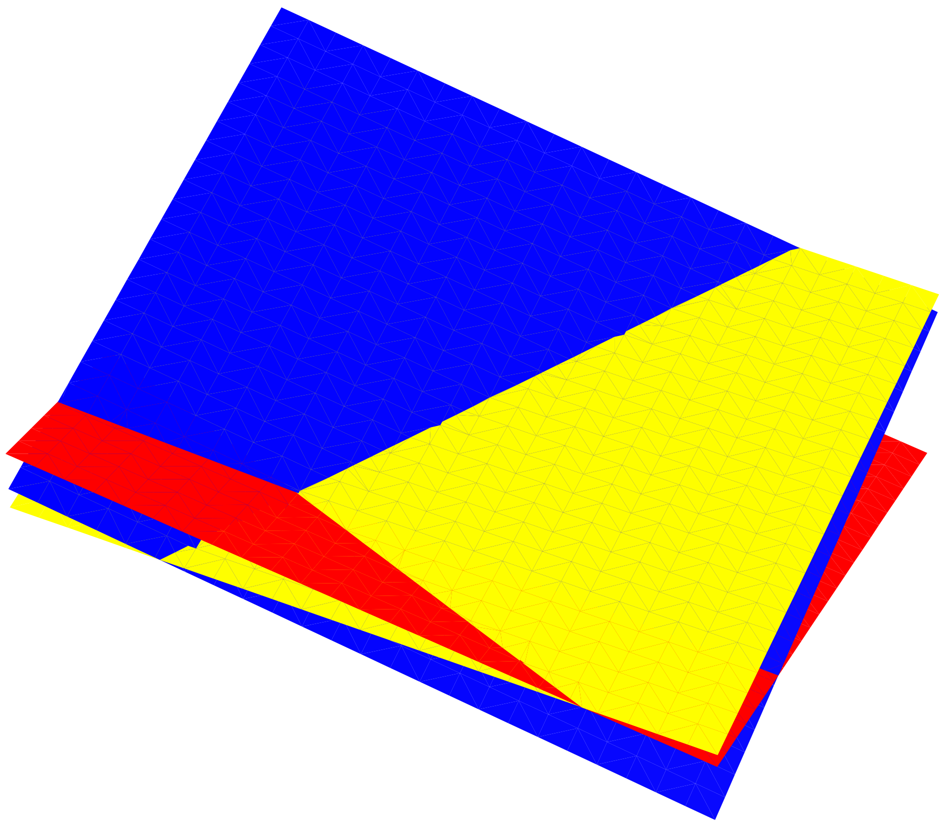
$$\boxed{\operatorname{rank} A \le \operatorname{rank} \tilde{A} \le \operatorname{rank} A + 1}$$

One-Point Intersection

$$2x + 3y - z = -3$$
$$-x + 2y + 2z = 1$$
$$x + y - z = -2$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 3$$

One-Point Intersection



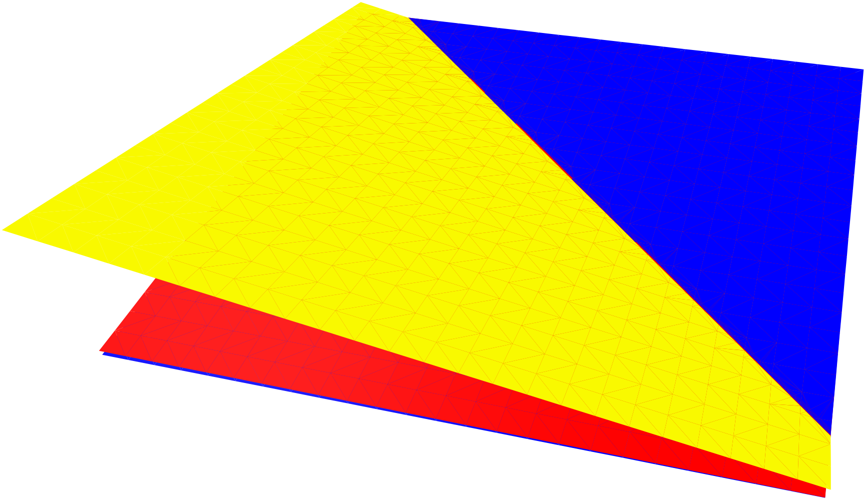
$$x = 1, y = -1, z = 2$$

One-Line Intersection

$$\begin{aligned} x - 2y - 3z &= 4 \\ 2x + 3y + z &= 1 \\ 3x - 4y - 7z &= 10 \end{aligned}$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2 < 3$$

One-Line Intersection



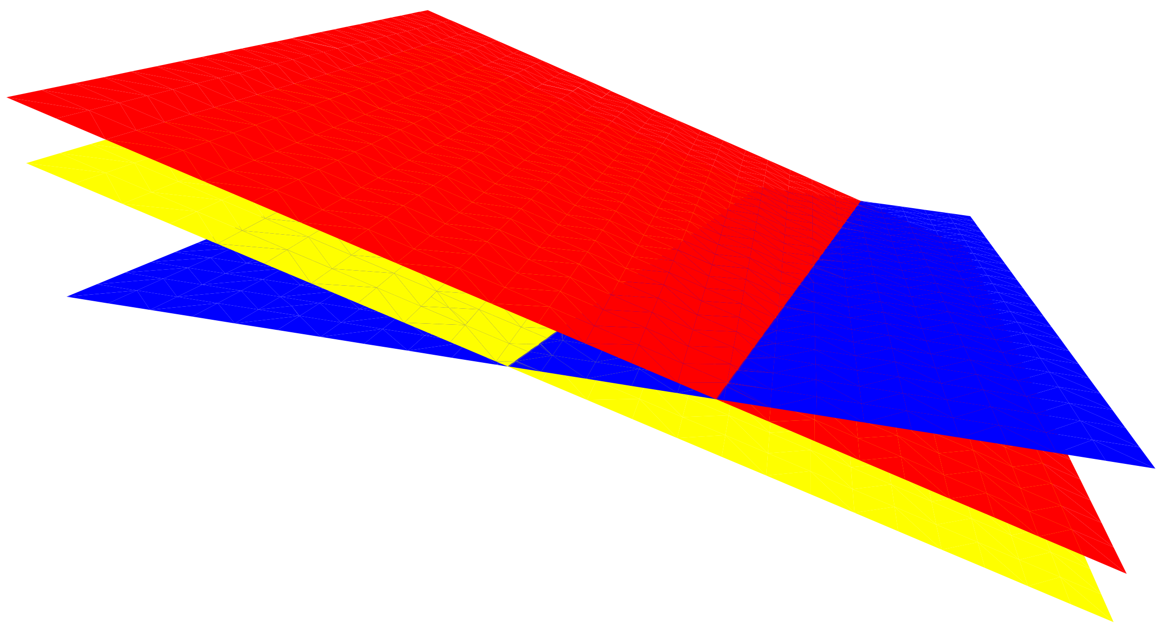
$$x = 2 + t,\; y = -1 - t,\; z = t$$

Parallel Two-Lines Intersection

$$
\begin{aligned}
x - 2y - 3z &= 4 \\
2x + 3y + z &= 4 \\
3x - 4y - 7z &= 10
\end{aligned}
$$

$$\operatorname{rank} A = 2 < \operatorname{rank} \tilde{A} = 3$$

Parallel Two-Lines Intersection

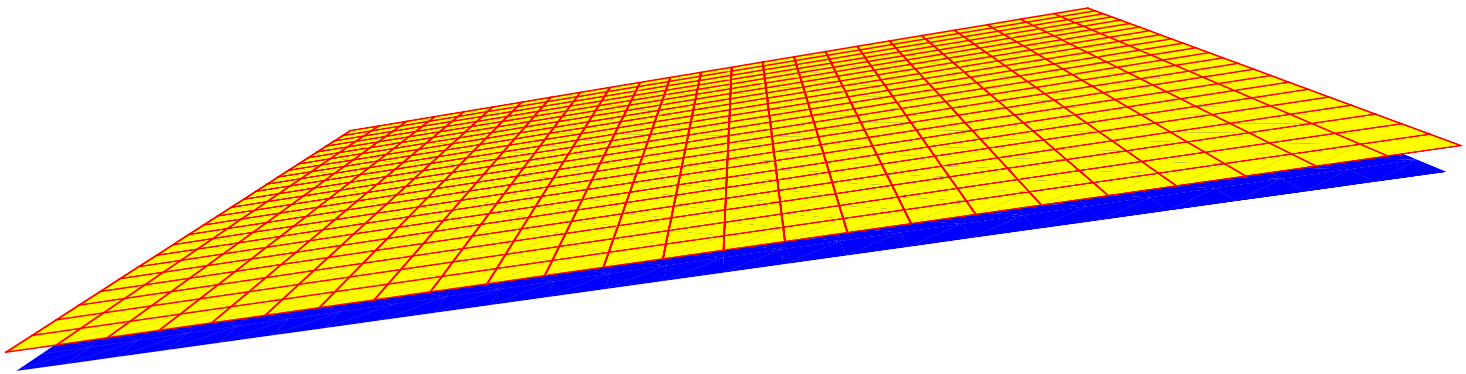


Parallel Two Planes

$$x - y + 3z = 1$$
$$3x - 3y + 9z = 3$$
$$x - y + 3z = 0$$

$$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$$

Parallel Two Planes

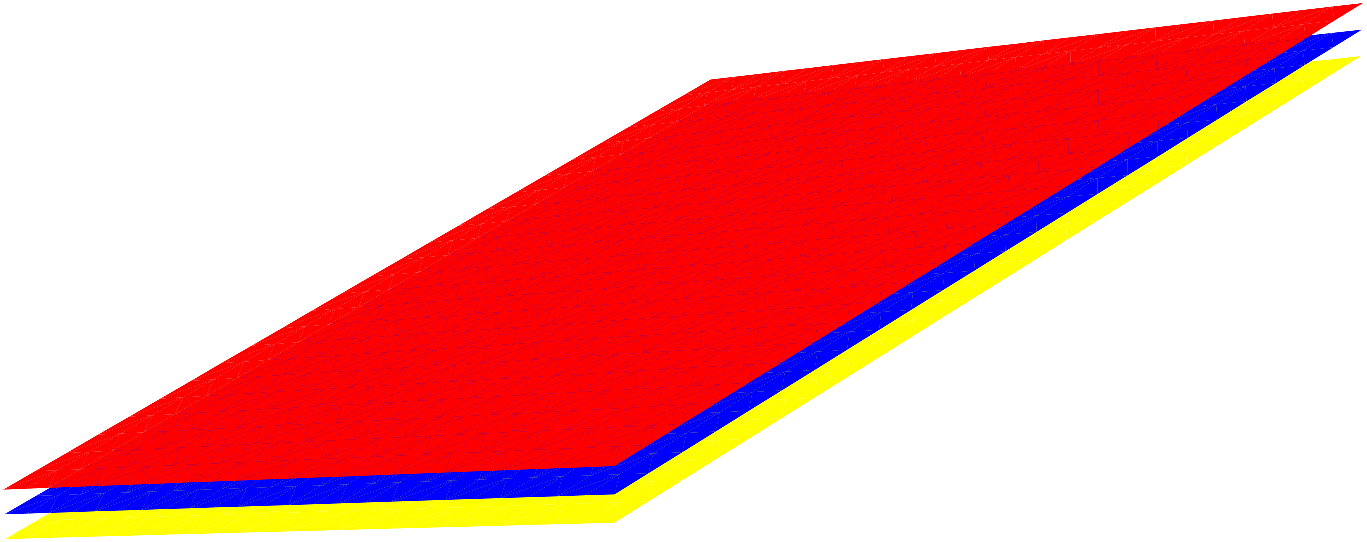


Parallel Three Planes

$$\boxed{\begin{aligned} x + 2y + 3z &= 10 \\ x + 2y + 3z &= 20 \\ x + 2y + 3z &= 30 \end{aligned}}$$

$$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$$

Parallel Three Planes



Superposed Three Planes

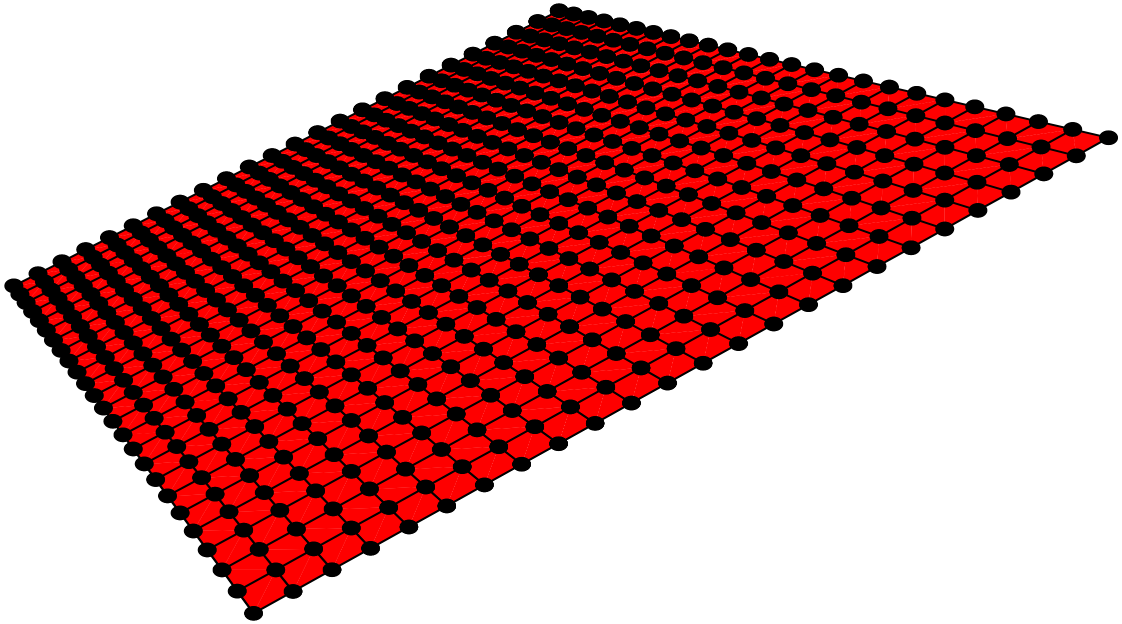
$$x + 2y + 3z = 20$$

$$2x + 4y + 6z = 40$$

$$3x + 6y + 9z = 60$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 3$$

Superposed Three Planes



System of Linear Equations

General Form

$$\boxed{\sum_{j=1}^{n} a_{ij} x_j = b_i}$$

Direct Solution

$$\begin{aligned} 3x_2 - 2x_3 + 3x_4 &= -4 \\ x_1 + x_2 + 3x_3 + 2x_4 &= 2 \\ x_1 + 2x_2 + 2x_3 + 3x_4 &= 1 \\ x_1 + 3x_2 + 2x_3 + 4x_4 &= -1 \end{aligned}$$

$\Downarrow$ **Transformation of Equations**

$$\begin{aligned} x_1 + x_4 &= 7 \\ x_2 + x_4 &= -2 \\ x_3 &= -1 \\ x_4 &= \alpha \end{aligned}$$

(Indefinite)

Gaussian Sweeping Out

Original Form

$$\boxed{\sum_{j=1}^{n} a_{ij} x_j = b_i}$$

Idea of Gauss

$A\mathbf{x} = \mathbf{b}$

$J\mathbf{x} = \mathbf{c}$

Matrix Representation

Original Form

$$\tilde{A} = (A, \mathbf{b}) \quad \Rightarrow \quad \tilde{J} = (J, \mathbf{c})$$

Left Elemntary Transformations

Matrix Representation (1)

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

Matrix Representation (2)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix}$$

Matrix Representation (3)

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

Matrix Representation (4)

$$\boxed{\sum_{j=1}^{n} a_{ij} x_j = b_i}$$

$$\Downarrow$$

$$\boxed{A\mathbf{x} = \mathbf{b}}$$

Coefficient Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} & b_n \end{pmatrix}$$

Left Elementary Transformations

(1) Interchange two rows
(2) Multiply a row by a non-zero constant
(3) Add a row by a multiplied another row

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & \cdot & 0 & c_{1r+1} & \cdots & c_{1n} & d_1 \\ 0 & 1 & \cdot & \cdot & c_{2r+1} & \cdots & c_{2n} & d_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & c_{rr+1} & \cdots & c_{rn} & d_r \\ 0 & 0 & \cdot & \cdot & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Equation after Left Elementary Transformations

$$\begin{cases} x_1 + c_{1r+1}x_{r+1} + \cdots + c_{1n}x_n = d_1 \\ x_2 + c_{2r+1}x_{r+1} + \cdots + c_{2n}x_n = d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_r + c_{rr+1}x_{r+1} + \cdots + c_{rn}x_n = d_r \end{cases}$$

$$\begin{cases} 0x_1 + \cdots + 0x_r + 0x_{r+1} + \cdots + 0x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0x_1 + \cdots + 0x_r + 0x_{r+1} + \cdots + 0x_n = 0 \end{cases}$$

Examples

Example (n=3)

$$x_1 + 2x_2 - x_3 = -1$$

$$2x_1 + 4x_2 - x_3 = -1$$

$$x_1 + 3x_2 + x_3 = 2$$

Coefficient Matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ 1 & 3 & 1 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & -1 \\ 1 & 3 & 1 & 2 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 3$$

Equation after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Unique Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Example (n=4)

$$
\begin{aligned}
3x_2 - 2x_3 + 3x_4 &= -4 \\
x_1 + x_2 + 3x_3 + 2x_4 &= 2 \\
x_1 + 2x_2 + 2x_3 + 3x_4 &= 1 \\
x_1 + 3x_2 + 2x_3 + 4x_4 &= -1
\end{aligned}
$$

General Solution

$$x_1 + x_4 = 7$$

$$x_2 + x_4 = -2$$

$$x_3 = -1$$

$$x_4 = \alpha$$ **(Indefinite)**

Coefficient Matrix

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 3 < 4$$

Equation after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -1 \\ 0 \end{pmatrix}$$

$$x_1 + x_4 = 7$$

$$x_2 + x_4 = -2$$

$$x_3 = -1$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

General Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Recurrence Formula for Sequences

Matrix Form

$$x_n = 4x_{n-1} + 10y_{n-1}, \quad x_0 = 3$$

$$y_n = -3x_{n-1} - 7y_{n-1}, \quad y_0 = 1$$

$$\Longrightarrow$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -3 & -7 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

Linea Algebra
and
Differential Equations

System of Differential Equations

Linear Case

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

Matrix Form

$$U(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow$$

$$\frac{dU}{dt} = AU(t)$$

Exponential Matrix

Main Idea

$$u''(t)+2bu'(t)+cu(t)=0$$

Matrix Representation

$$u''(t)+2bu'(t)+cu(t)=0$$

Original Form

$$\frac{dU(t)}{dt}=AU(t) \Rightarrow \textbf{Calculation of } e^{tA}$$

Solution (1)

$$\begin{cases} u_1(t) = u(t), \\ u_2(t) = u'(t) \end{cases}$$

$$\begin{cases} u_1'(t) = u'(t) = u_2(t), \\ u_2'(t) = u''(t) = -2bu'(t) - cu(t) \\ \qquad\quad = -2bu_2(t) - cu_1(t) \end{cases}$$

Solution (2)

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}$$

Solution (3)

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\begin{cases} \dfrac{d}{dt} U(t) = AU(t), \\ U(0) = U_0 \end{cases}$$

Solution (4)

$$\boxed{U(t) = e^{tA} U_0}$$

$$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots$$

(**Exponential Matrix**)

Example
of
Exponential Matrices

Simple Eigenvalue Case

Example

$$\frac{dx}{dt} = 4x + 10y, \qquad x(0) = 3$$

$$\frac{dy}{dt} = -3x - 7y, \qquad y(0) = 1$$

Matrix Form

$$U(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A := \begin{pmatrix} 4 & 10 \\ -3 & -7 \end{pmatrix}$$

$$\Rightarrow$$

$$\frac{dU}{dt} = AU(t)$$

Diagonalization

$$A = \begin{pmatrix} 4 & 10 \\ -3 & -7 \end{pmatrix}, P = \begin{pmatrix} -2 & -5 \\ 1 & 3 \end{pmatrix}$$

$$\Rightarrow$$

$$P^{-1}AP = \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Reduction (1)

$$\boxed{\begin{aligned} V(t) &= \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \\ &= P^{-1}U(t) \\ &= \begin{pmatrix} -3 & -5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \end{aligned}}$$

Reduction (2)

$$\frac{d\mathbf{V}}{dt} = P^{-1}\frac{dU}{dt} = P^{-1}AU(t)$$

$$= \left(P^{-1}AP\right)\mathrm{V}(t) = \Lambda \mathrm{V}(t)$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} V(t)$$

Reduction (3)

$$\begin{cases} \dfrac{dz}{dt} = -z \\ \dfrac{dw}{dt} = -2w \end{cases}$$

$$\begin{pmatrix} z(0) \\ w(0) \end{pmatrix} = P^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -14 \\ 5 \end{pmatrix}$$

Reduction (4)

$$\begin{cases} \dfrac{dz}{dt} = -z, & z(0) = -14 \\ \dfrac{dw}{dt} = -2w, & w(0) = 5 \end{cases}$$

$$\Rightarrow$$

$$\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} -14e^{-t} \\ 5e^{-2t} \end{pmatrix}$$

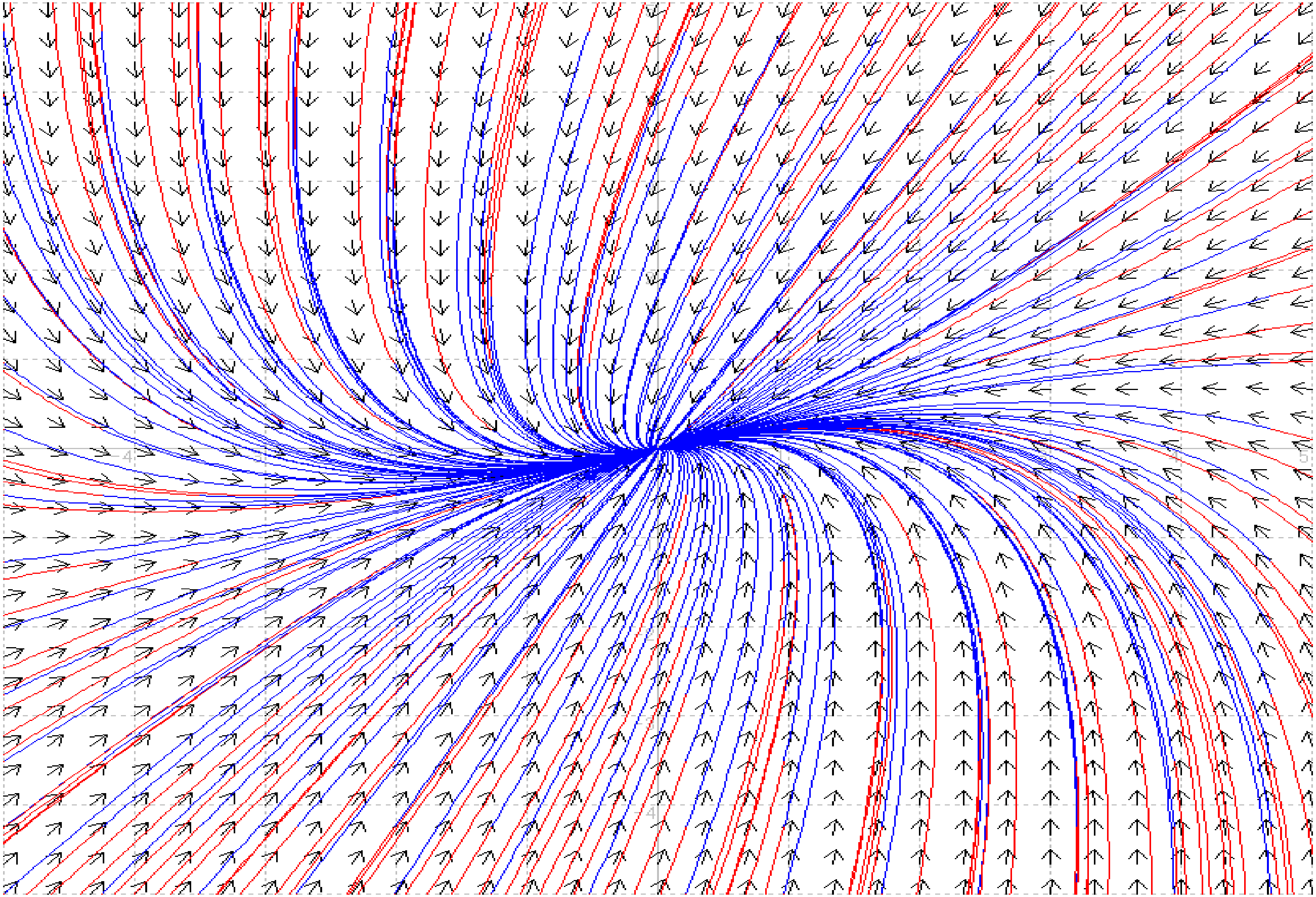
Solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -14e^{-t} \\ 5e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} 28e^{-t} - 25e^{-2t} \\ -14e^{-t} + 15e^{-2t} \end{pmatrix}$$

Stable Node



Double Eigenvalue Case

Jordan Canonical Form

of

Matrices

Marie Ennemond Camille Jordan

Jordan

- **Marie Ennemond Camille Jordan (1838-1922)**

French Mathematician

Jordan's Canonical Form

$$P^{-1}AP = \Lambda \quad \textbf{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Calculation (1)

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}}$$

$$\boxed{|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c}$$

Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c = 0$$

$$\lambda = -b \quad \textbf{(Double Root)}$$

$$P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

Calculation (3)

$$P^{-1}AP = \Lambda \quad \textbf{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -b & 1 \\ 0 & -b \end{pmatrix}$$

Calculation (4)

$$
\begin{aligned}
& P^{-1} e^{tA} P \\
&= P^{-1}\left(I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right) P \\
&= P^{-1}P + t\left(P^{-1}AP\right) + \frac{t^2}{2!}\left(P^{-1}AP\right)\left(P^{-1}AP\right) + \cdots + \\
&\qquad + \frac{t^n}{n!}\underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}_{n-\text{times}} + \cdots \\
&= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots \\
&= e^{t\Lambda}
\end{aligned}
$$

Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!}\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Calculation (6)

$$e^{tA} = Pe^{t\Lambda}P^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} - \lambda te^{\lambda t} & te^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1)e^{\lambda t} \end{pmatrix}$$

Calculation (7)

$$\textbf{Case}: D/4 = b^2 - c = 0$$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} - \lambda t e^{\lambda t} & t e^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1) e^{\lambda t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

Canonical Forms of Quadratic Forms

Quadratic Form of Two Variables

$$\boxed{\begin{aligned} z &= f(x, y) \\ &= ax^2 + 2bxy + cy^2 \end{aligned}}$$

Matrix Form

$$z = f(x, y)$$

$$= ax^2 + 2bxy + cy^2$$

$$\Longrightarrow$$

$$ax^2 + 2bxy + cy^2$$

$$= \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Example 1

$$z = f(x, y)$$

$$= 3x^2 - 2xy + 3y^2$$

$$\Rightarrow$$

$$3x^2 - 2xy + 3y^2$$

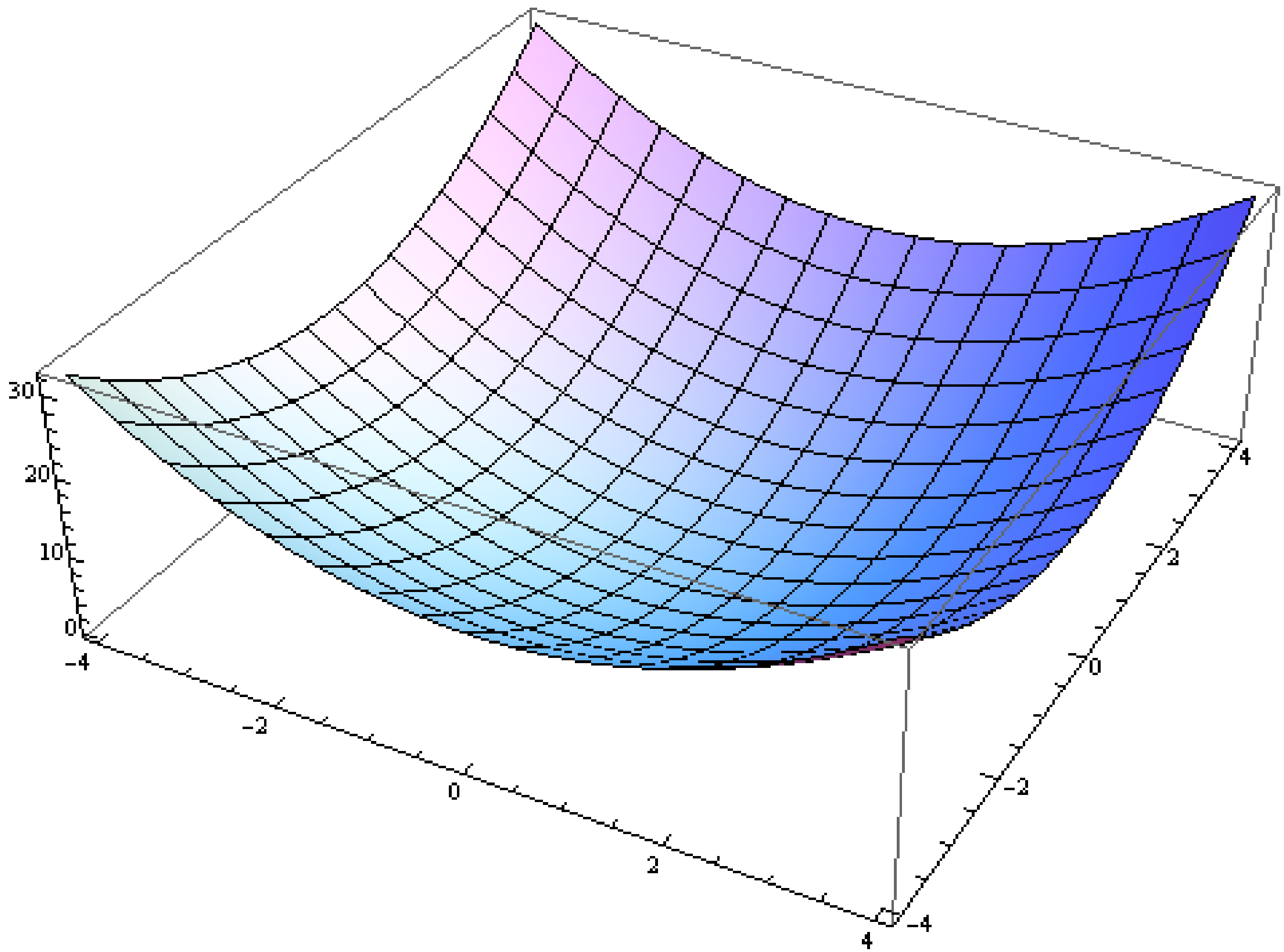
$$= \left\langle \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

Eigenvalues: 2, 4

$$z = x^2 + 2y^2 \quad \textbf{(ellipse)}$$



Example 2

$$z = f(x, y)$$

$$= x^2 - 6xy + y^2$$

$$\Rightarrow$$

$$x^2 - 6xy + y^2$$

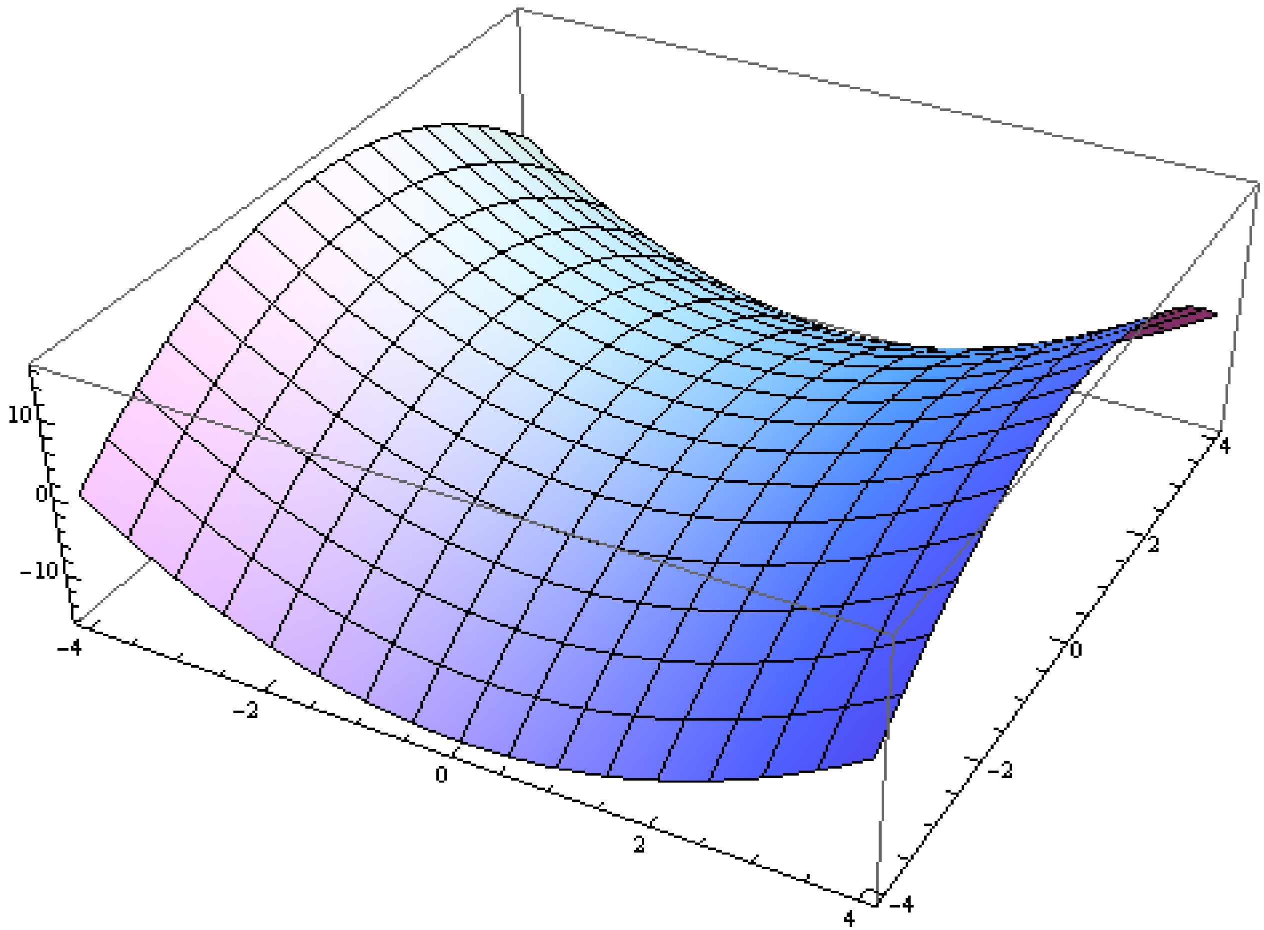
$$= \left\langle \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

Eigenvalues : - 2, 4

$z = -2x^2 + 4y^2$ **(hyperbola)**



Example 3

$$z = f(x, y)$$

$$= 4x^2 - 4xy + y^2$$

$$\Rightarrow$$

$$4x^2 - 4xy + y^2$$

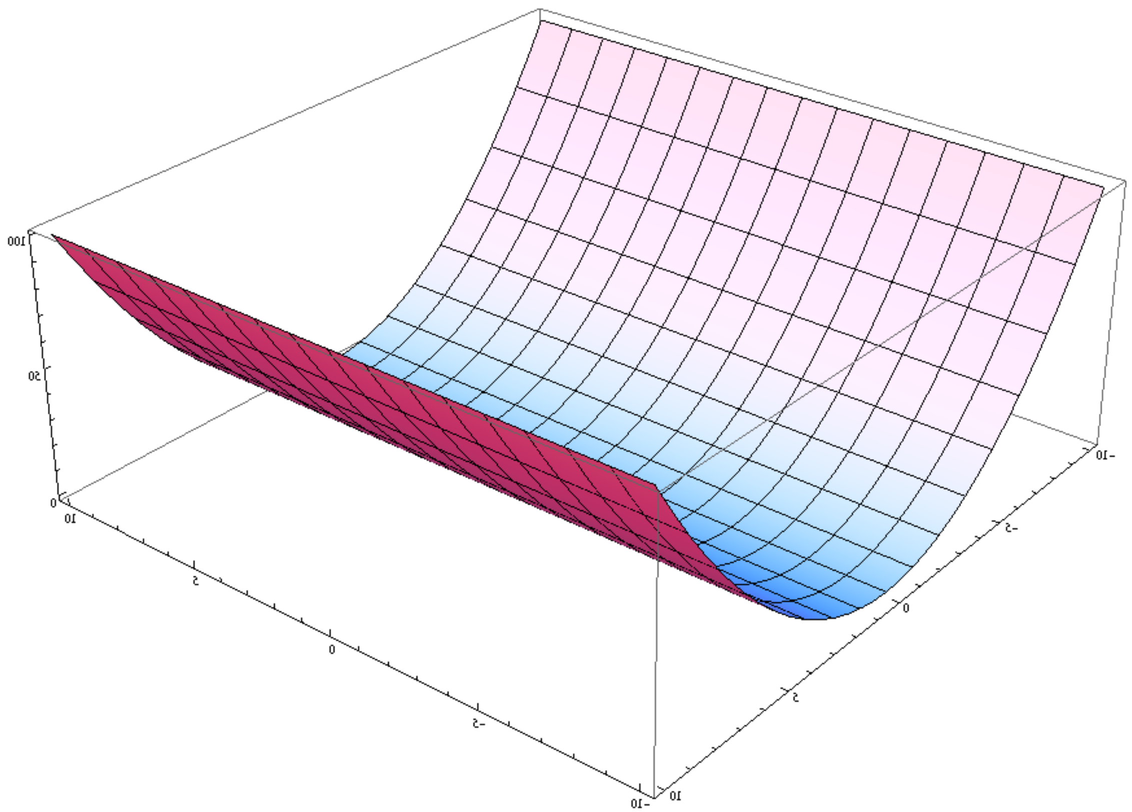
$$= \left\langle \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{Eigenvalues:}\ \mathbf{0,\ \ 5}$$

$$z = 5y^2 + \sqrt{5}x$$ **(parabola)**



Linea Algebra
and
Differential Equations

2-dimensional Autonomous System

Linear Case

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

Matrix Form

$$U(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow$$

$$\frac{d}{dt} U(t) = A U(t)$$

Stability of Solutions

Computational Approach

Numerical Computing
with
BASIC

Example 1 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x \\[2ex] \dfrac{dy}{dt} = y \end{cases}$$

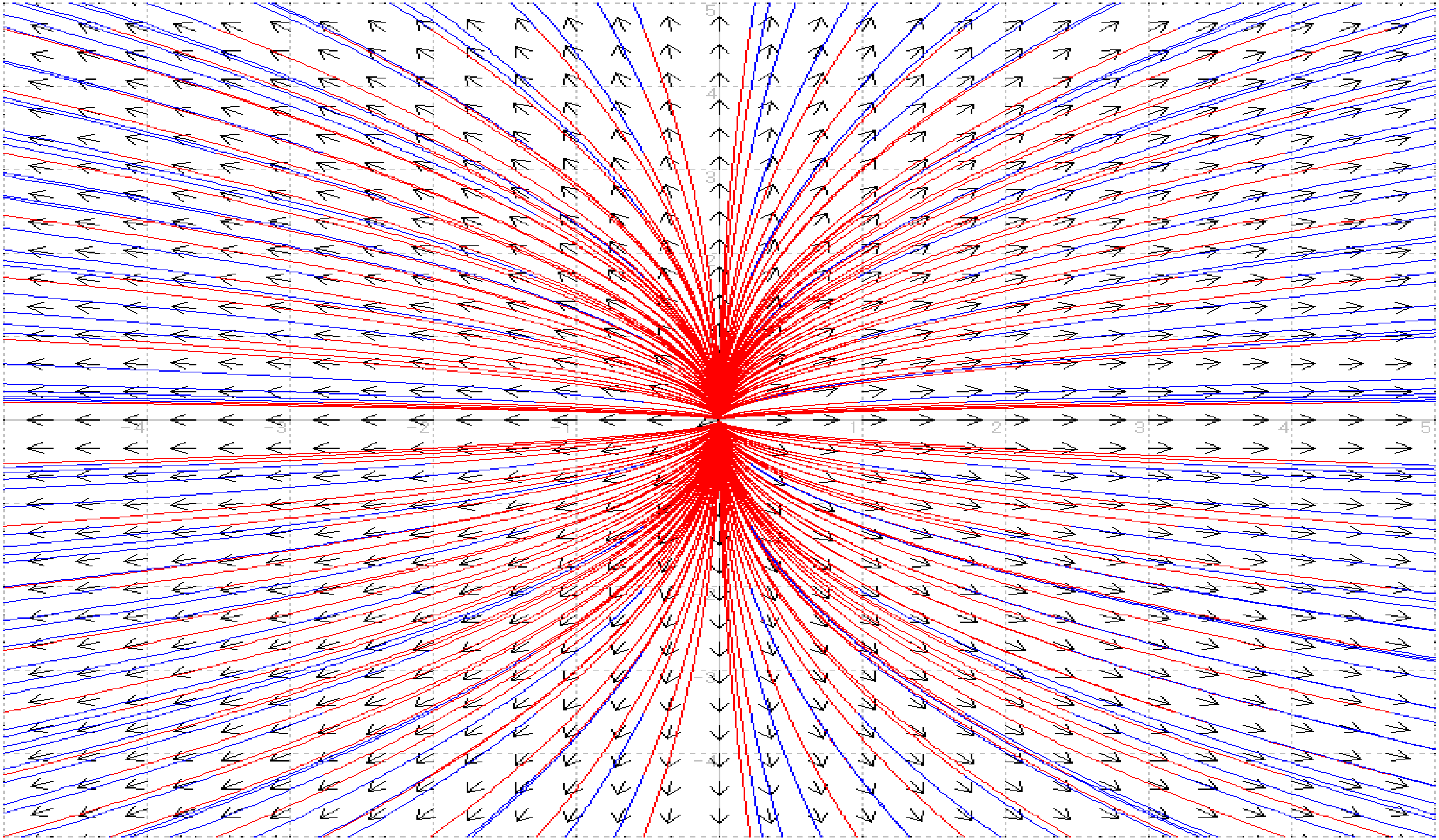
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 2,\ 1$$

Unstable Node



Example 2 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y \end{cases}$$

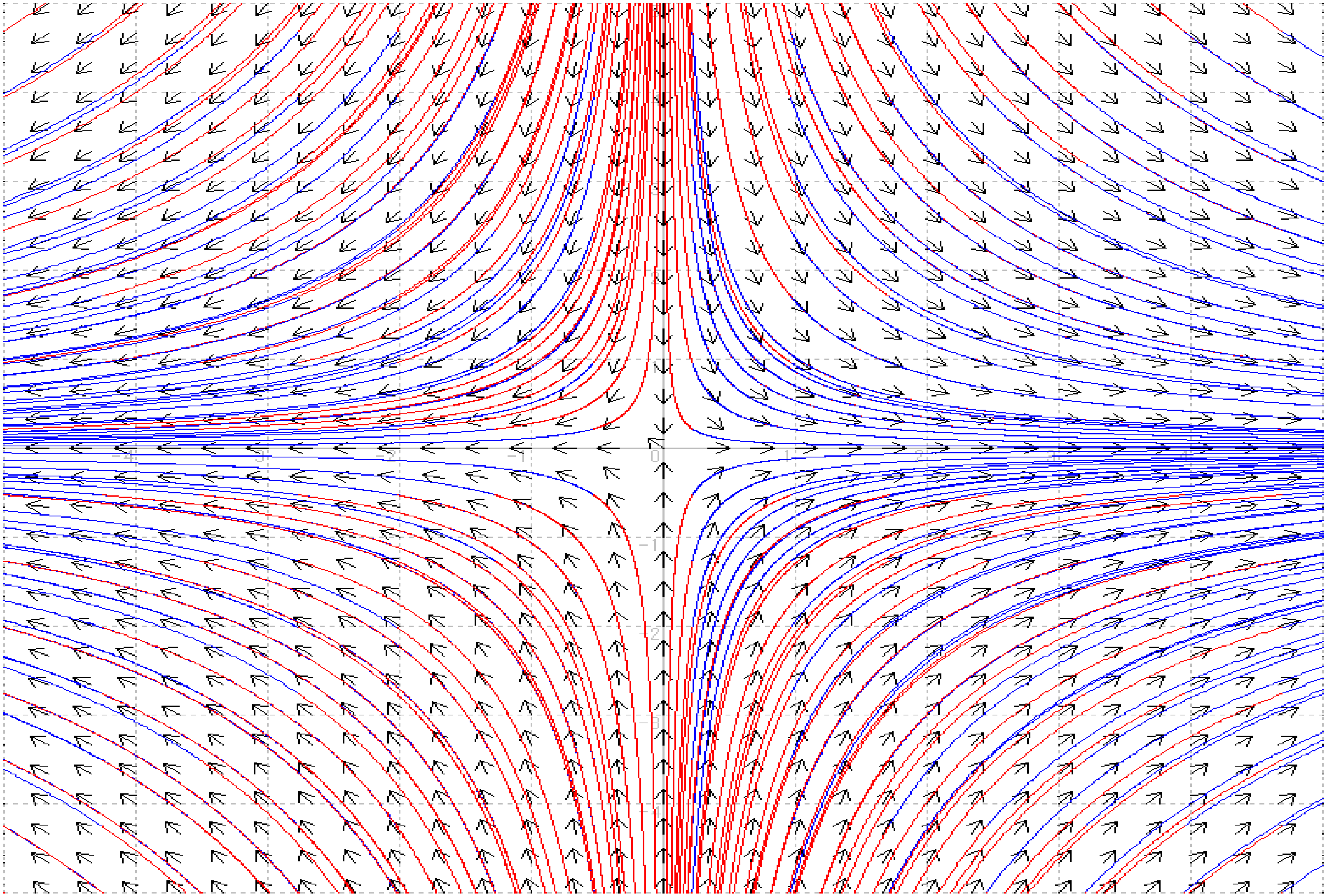
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 1, \quad -1$$

Saddle Point



Example 3 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$$

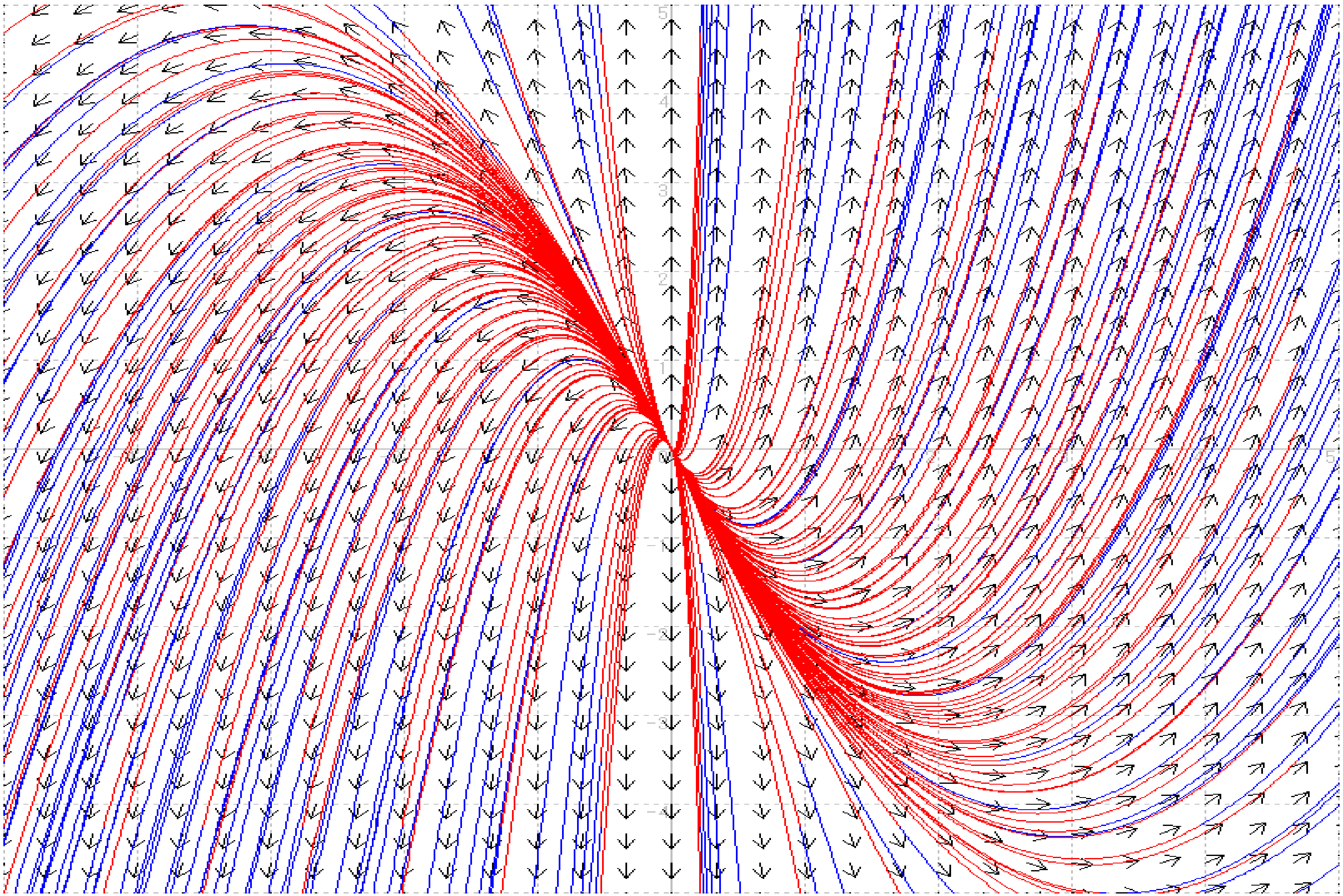
$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 1,\ \ 2$$

Unstable Node



Example 4 (Stable Node)

$$\begin{cases} \dfrac{dx}{dt} = -2x - 1.5y \\[2ex] \dfrac{dy}{dt} = x - 5.5y \end{cases}$$

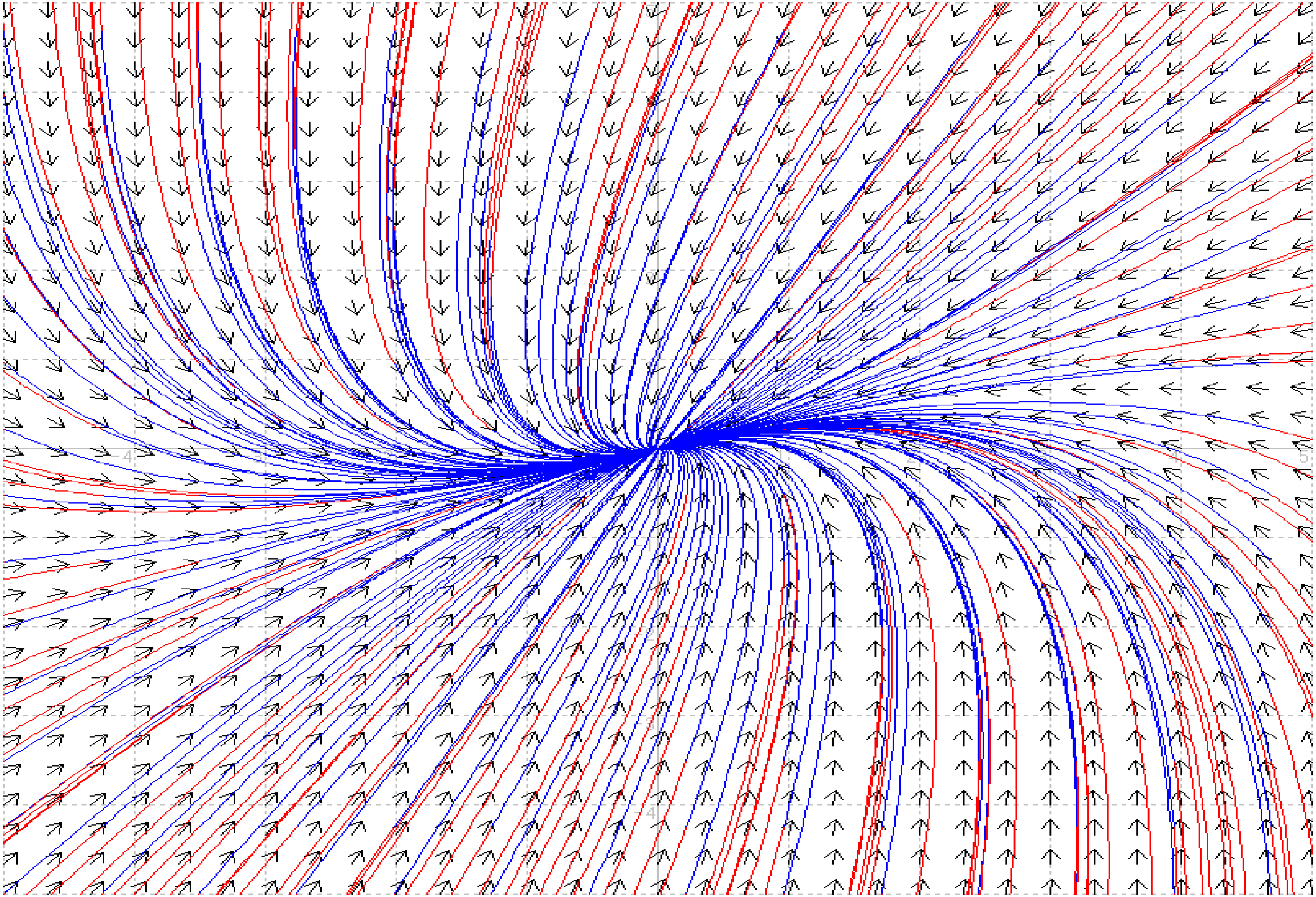
$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

$$\textbf{Eigenvalues}: -2.5,\ \ -5$$

Stable Node



Example 5 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = -2x + 2y \\ \dfrac{dy}{dt} = -2x + 3y \end{cases}$$

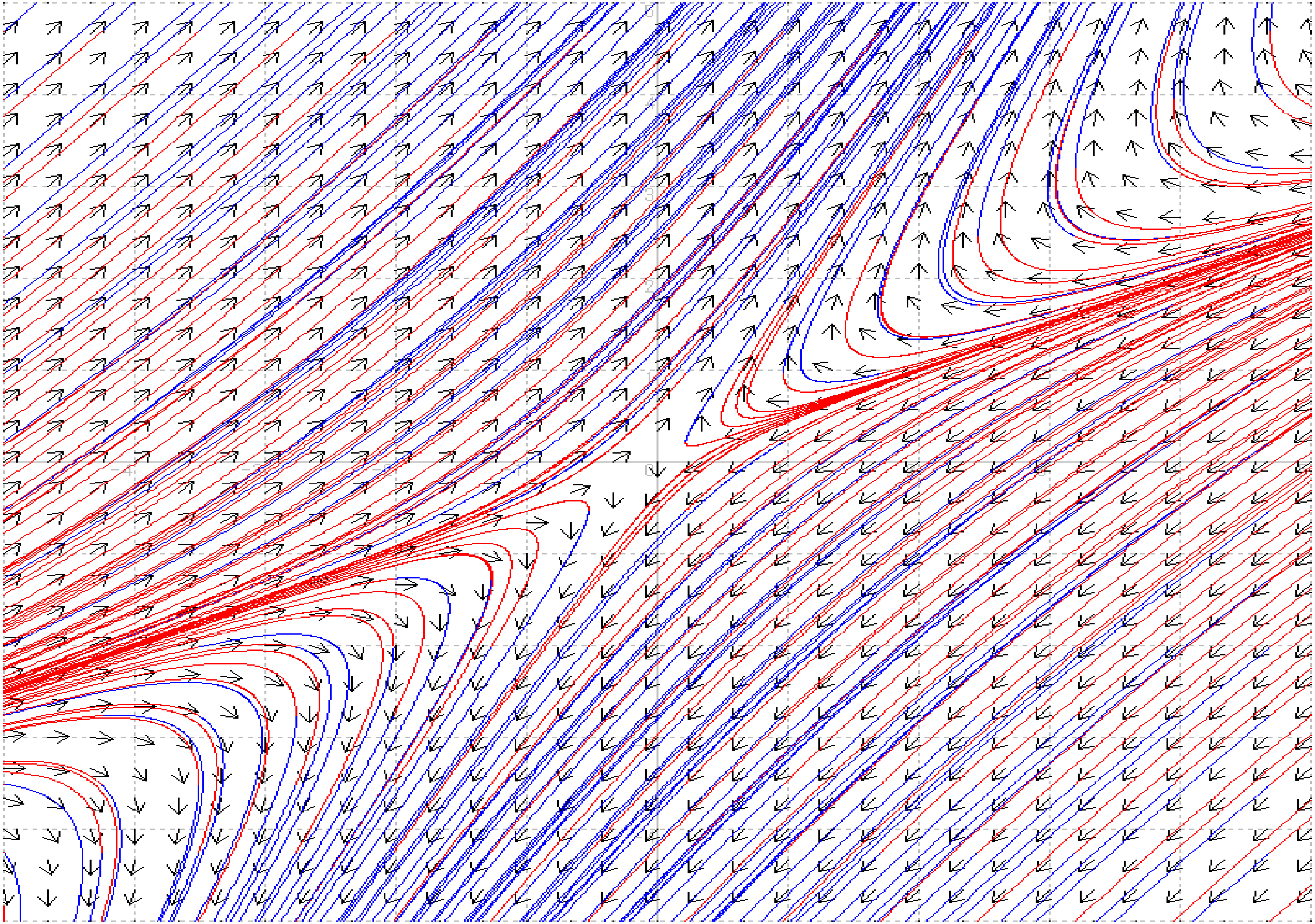
$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 2, \quad -1$$

Saddle Point



Example 6 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + y \\ \dfrac{dy}{dt} = 2y \end{cases}$$

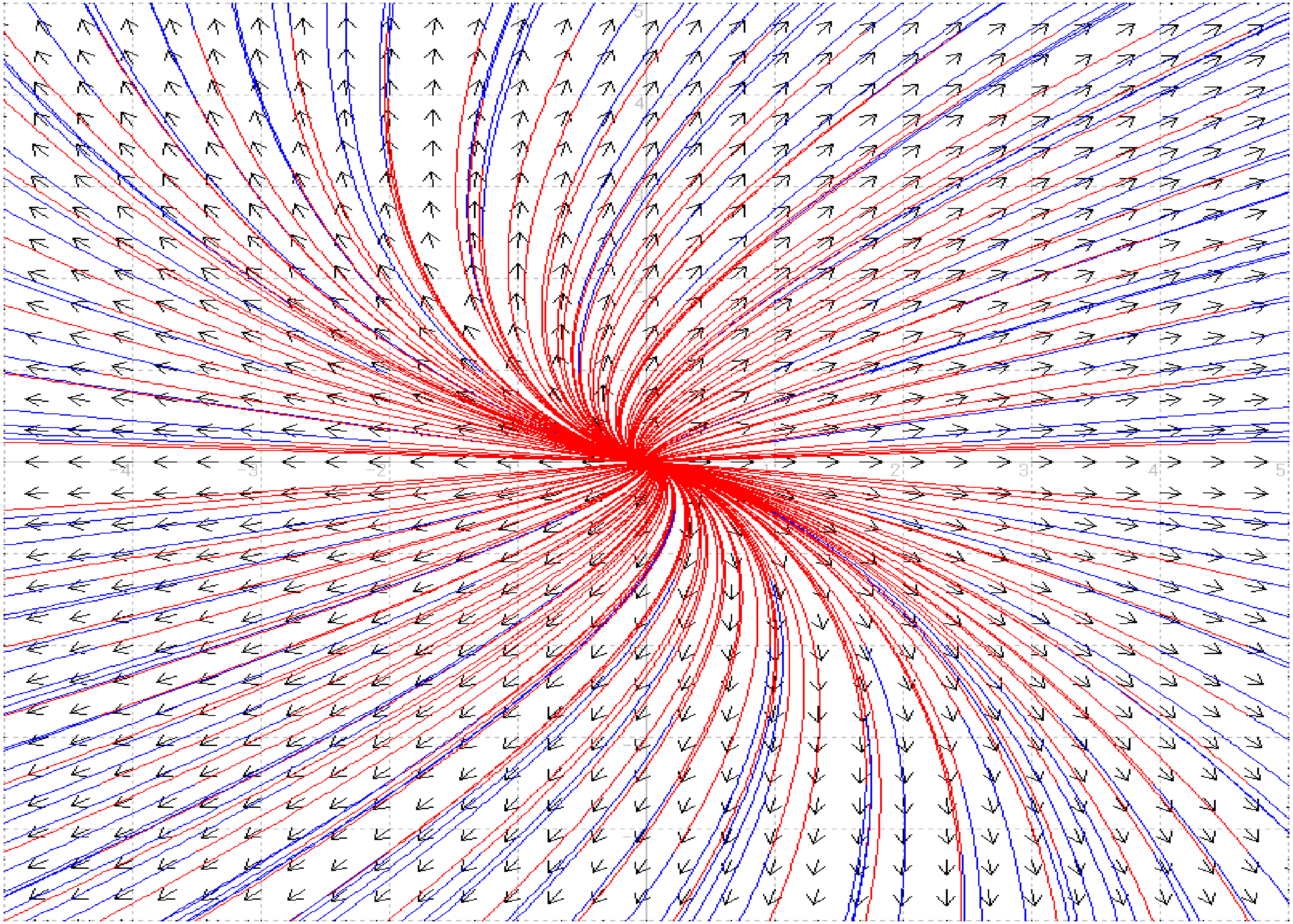
$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 2, \quad 2$$

Unstable Node



Example 7 (Center)

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = -x - y \end{cases}$$

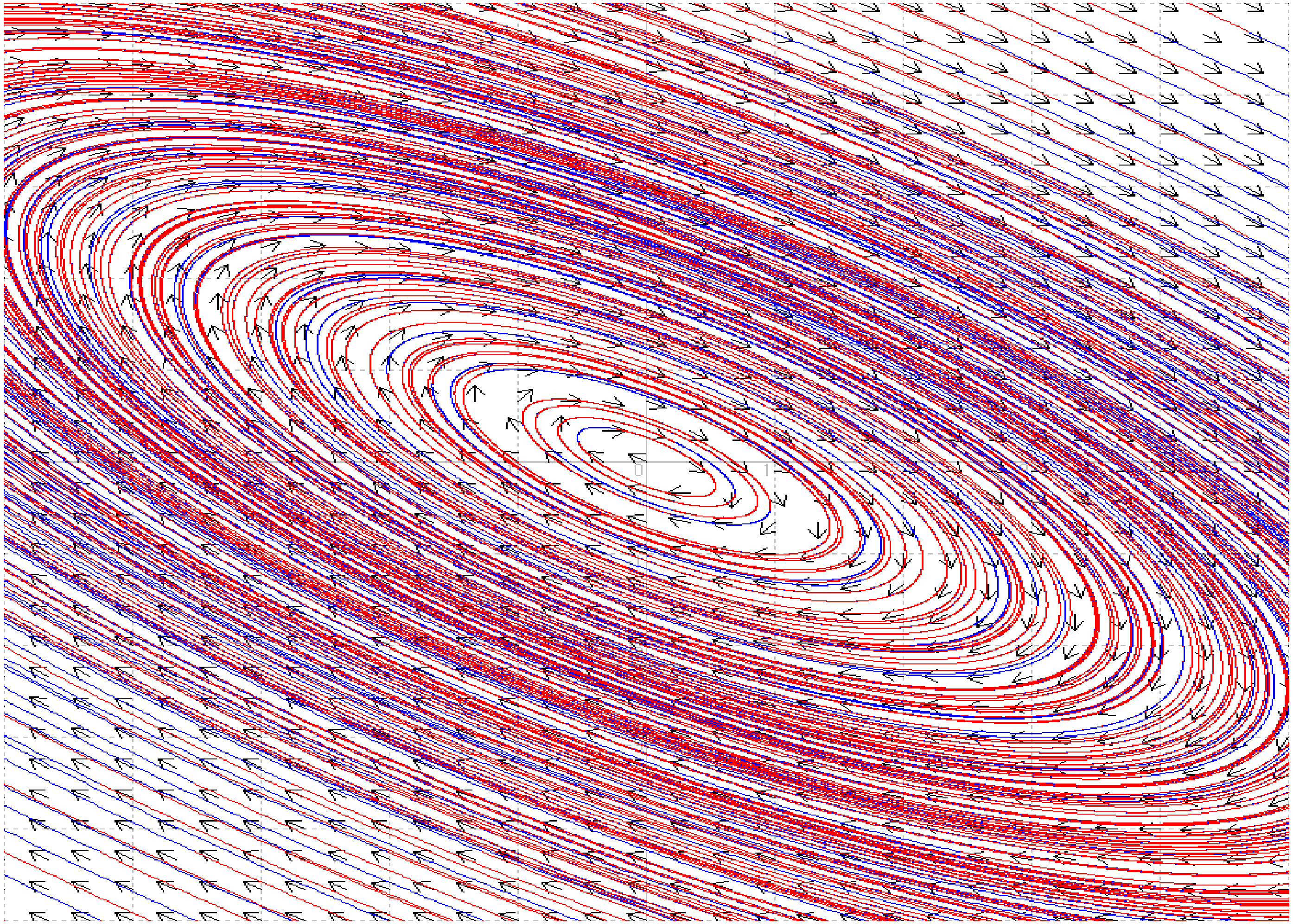
$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$\textbf{Eigenvalues}: \sqrt{-1}, \; -\sqrt{-1}$$

Center



Example 8 (Unstable Focus)

$$\begin{cases} \dfrac{dx}{dt} = x + y \\[2ex] \dfrac{dy}{dt} = -2x + y \end{cases}$$

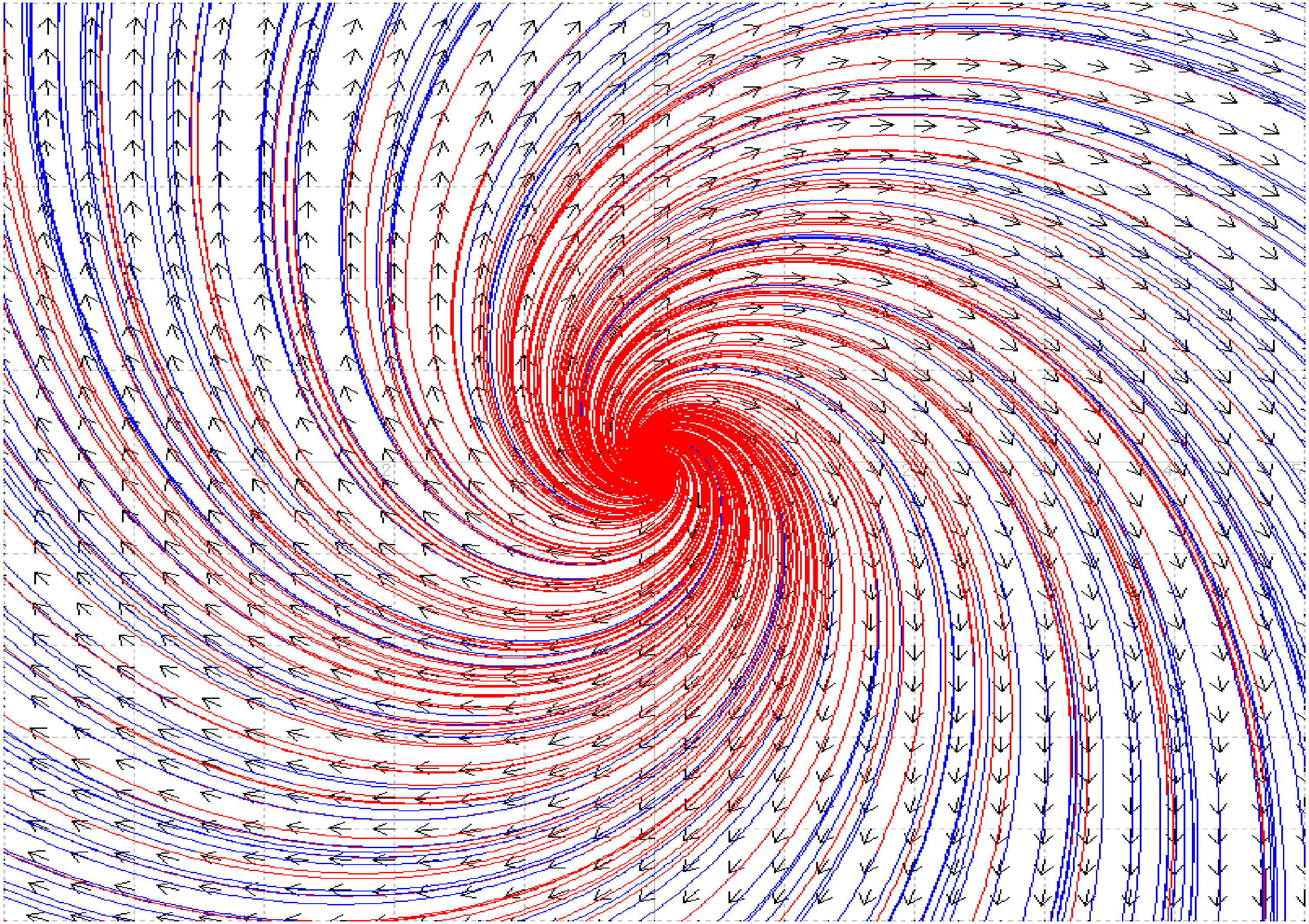
$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 1+\sqrt{2}i,\ 1-\sqrt{2}i$$

Unstable Node



Example 9 (Degenerate Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + 2y \\ \dfrac{dy}{dt} = 3x + 3y \end{cases}$$

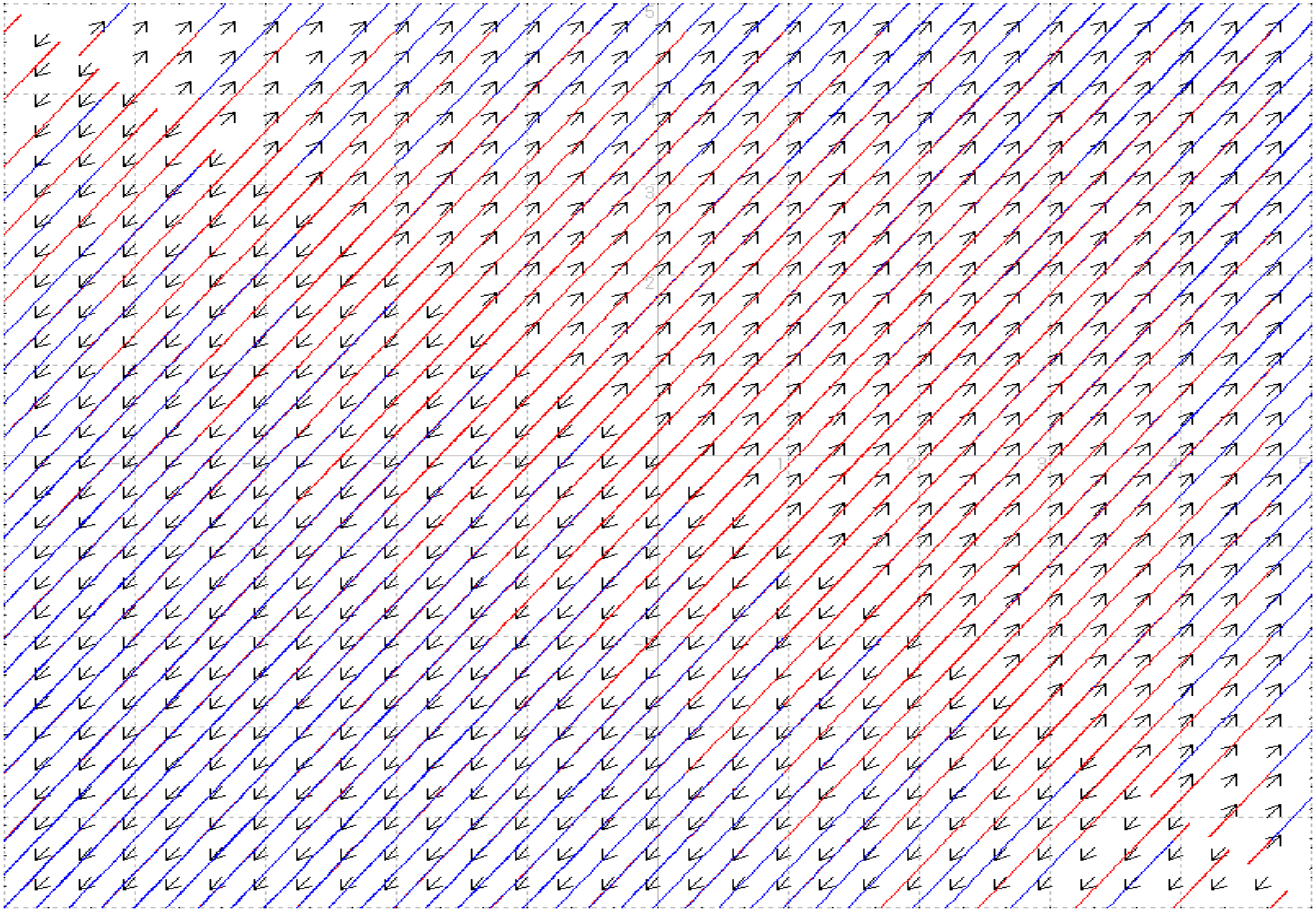
$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

$$\textbf{Eigenvalues}: \color{blue}{0}, \quad \color{red}{5}$$

Degenerate Node



Rank of Matrices Revisited

Definition of Rank

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix}$$

$\Longrightarrow$

Left Elemntary Transformations

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & \cdot & 0 & c_{1r+1} & \cdot\cdot & c_{1n} \\ 0 & 1 & \cdot & \cdot & c_{2r+1} & \cdot\cdot & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & \cdot & 1 & c_{rr+1} & \cdot\cdot & c_{rn} \\ 0 & 0 & \cdot & \cdot & 0 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \cdot\cdot & 0 \end{pmatrix}$$

$$\boxed{\operatorname{rank} A = \textbf{Number of } 1}$$

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation

Original Form

Placement of Lines and Planes

Examples

Computational Approach

Numerical Computing
with
BASIC

Example 1

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

2 行と 1 行を入れ替える

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{matrix}$$

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 2 \end{matrix}$$

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

4 行を-1 倍し，3 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 \\ 0 & 3 & -2 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Rank A = 3

Matrix after Left Elementary Transformations

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$\boxed{\operatorname{rank} A = 3}$$

Example 2

$$B = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

2 行と 1 行を入れ替える

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{matrix}$$

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 2 & -3 \end{matrix}$$

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{matrix}$$

4 行を-1 倍し，3 行の 1 倍を引く

$$\begin{matrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 3 & -2 & 3 & -4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Rank B = 3

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\operatorname{rank} B = 3}$$

Example 3

$$C = \begin{pmatrix} 1 & -2 & -3 & 4 \\ 2 & 3 & 1 & 1 \\ 3 & -4 & -7 & 10 \end{pmatrix}$$

2 行を 1 倍し，1 行の 2 倍を引く

3 行を 1 倍し，1 行の 3 倍を引く

$$\begin{matrix} 1 & -2 & -3 & 4 \\ 0 & 7 & 7 & -7 \\ 0 & 2 & 2 & -2 \end{matrix}$$

3 行を 7 倍し，2 行の 2 倍を引く

$$\begin{matrix} 1 & -2 & -3 & 4 \\ 0 & 7 & 7 & -7 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Rank C = 2

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\operatorname{rank} C = 2}$$

Example 4

$$D = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\operatorname{rank} D = 3}$$

System of Linear Equations and Ranks

General Form (n=2)

$$\boxed{\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}}$$

Matrix Representation

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$

Coefficient Matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \end{pmatrix}$$

Idea of Rank (1)

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

$$\Leftrightarrow$$

$$x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Idea of Rank (2)

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

$$\Leftrightarrow$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A}$$

Linear Algebra and Geometry

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation ↓ ↑ **Original Form**

Placement of Lines

Classification of Intersections

$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2$	**One-Point**
$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$	**Parallel Two Lines**
$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 2$	**Superposed Two Lines**

$$\boxed{\operatorname{rank} A \le \operatorname{rank} \tilde{A} \le \operatorname{rank} A + 1}$$

Equation of a Line

$$\boxed{ax + by = c}$$

One-Point Intersection

$$\boxed{\begin{aligned} 2x + 3y &= 3 \\ 3x - 8y &= 17 \end{aligned}}$$

Coefficient Matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -8 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

Unique Solution

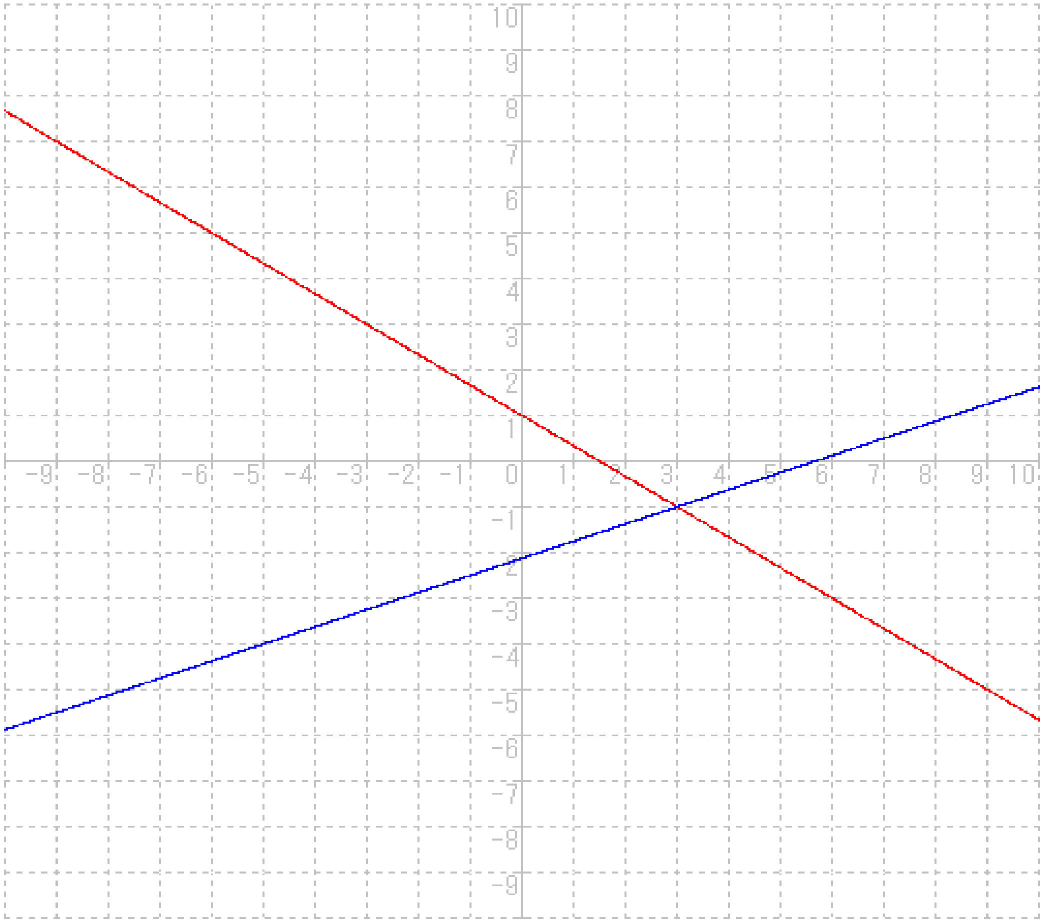
$$\tilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\boxed{\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2}$$

One-Point Intersection



$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2$$

Parallel Two Lines

$$x + 2y = 2$$

$$x + 2y = 3$$

Coefficient Matrix

$$\boldsymbol{A} = \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

No Solution

$$\widetilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

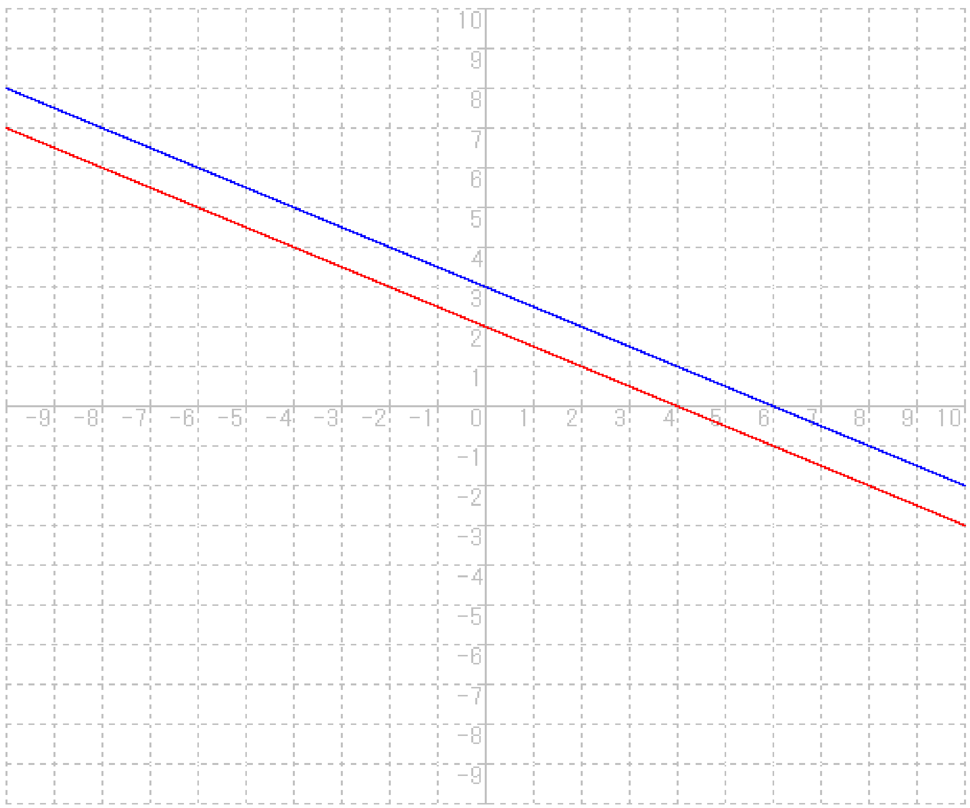
$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Impossible)

$$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$$

Parallel Two Lines



$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

Superposed Two Lines

$$\boxed{\begin{aligned} 6x - 2y &= -8 \\ 3x - y &= -4 \end{aligned}}$$

Coefficient Matrix

$$\boldsymbol{A} = \begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

Many Solutions

$$\tilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

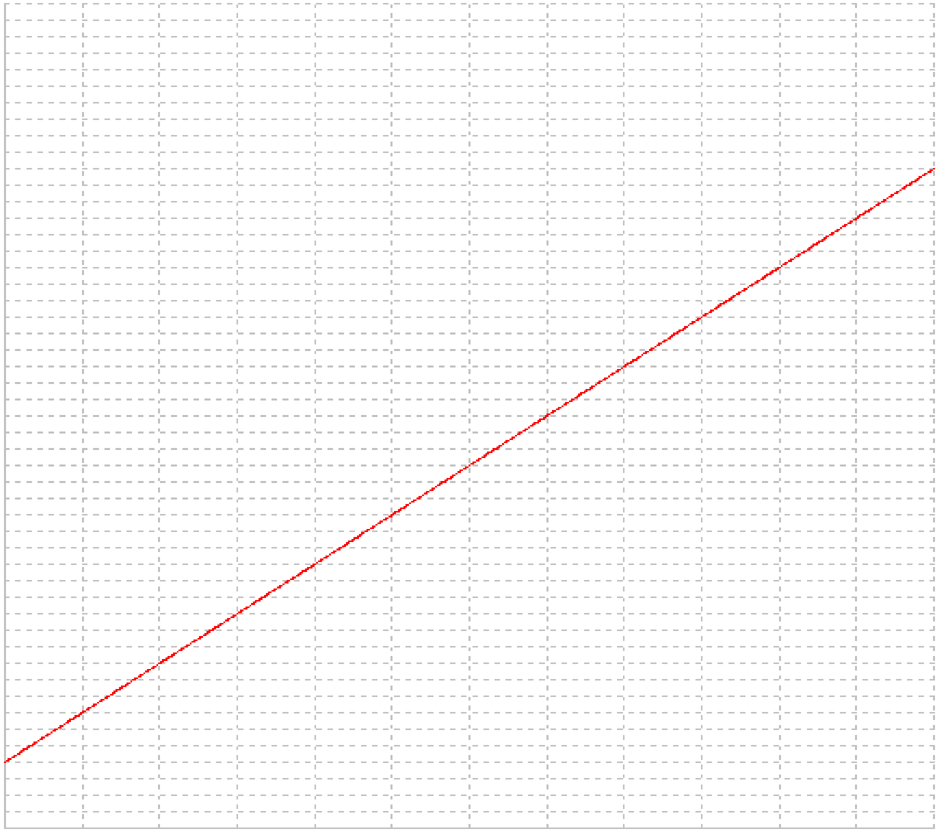
$$\Longrightarrow$$

$$\begin{pmatrix} 1 & -1/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

(Indefinite)

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 2$$

Superposed Two Lines



$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 2$$

General Form (n=3)

$$\boxed{\begin{aligned} ax + by + cz &= \alpha \\ dx + ey + fz &= \beta \\ gx + hy + kz &= \gamma \end{aligned}}$$

Matrix Representation

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Coefficient Matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a & b & c & \alpha \\ d & e & f & \beta \\ g & h & k & \gamma \end{pmatrix}$$

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation ↓

↑ **Original Form**

Placement of Planes

Classification of Intersections

$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 3$	**One-Point**
$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2 < 3$	**One Line**
$\operatorname{rank} A = 2 < \operatorname{rank} \tilde{A} = 3$	**Parallel Two Lines** **Parallel Three Lines**
$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 3$	**Superposed Three Planes**
$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$	**Parallel Two Planes** **Parallel Three Planes**

$$\boxed{\operatorname{rank} A \le \operatorname{rank} \tilde{A} \le \operatorname{rank} A + 1}$$

Equation of a Plane (1)

$$\boxed{ax + by + cz = d}$$

Equation of a Plane (2)

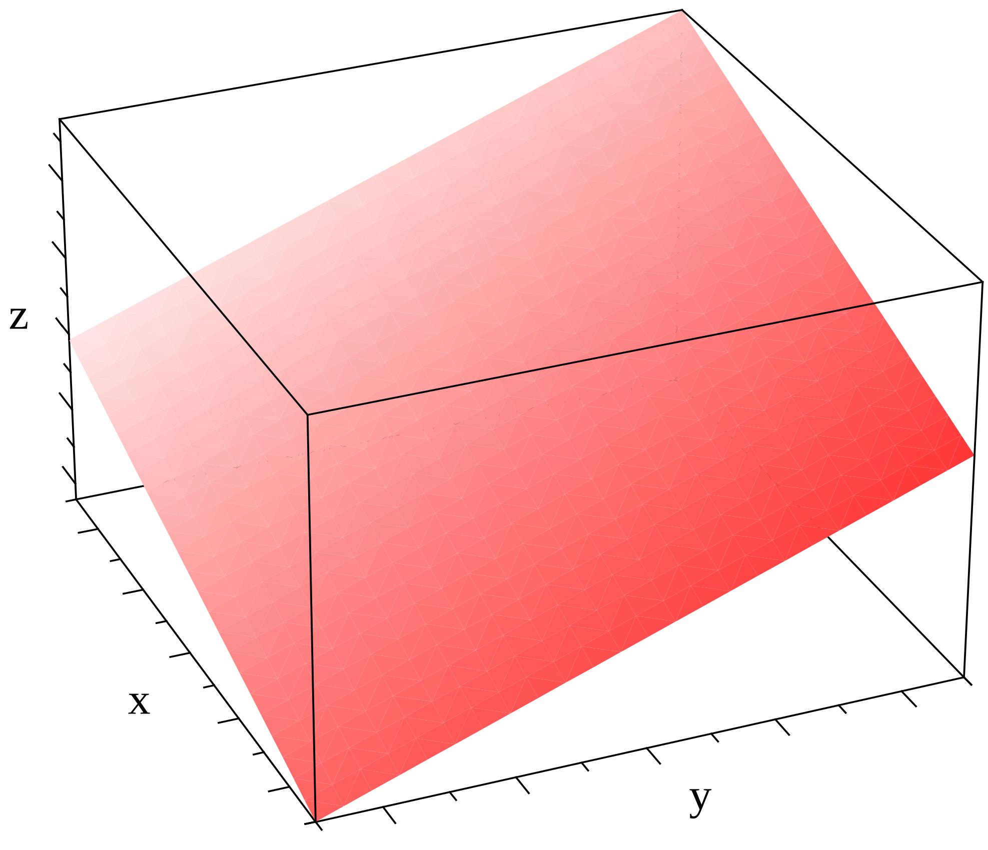
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Direction Vector

Numerical Computing with MuPAD

Plane

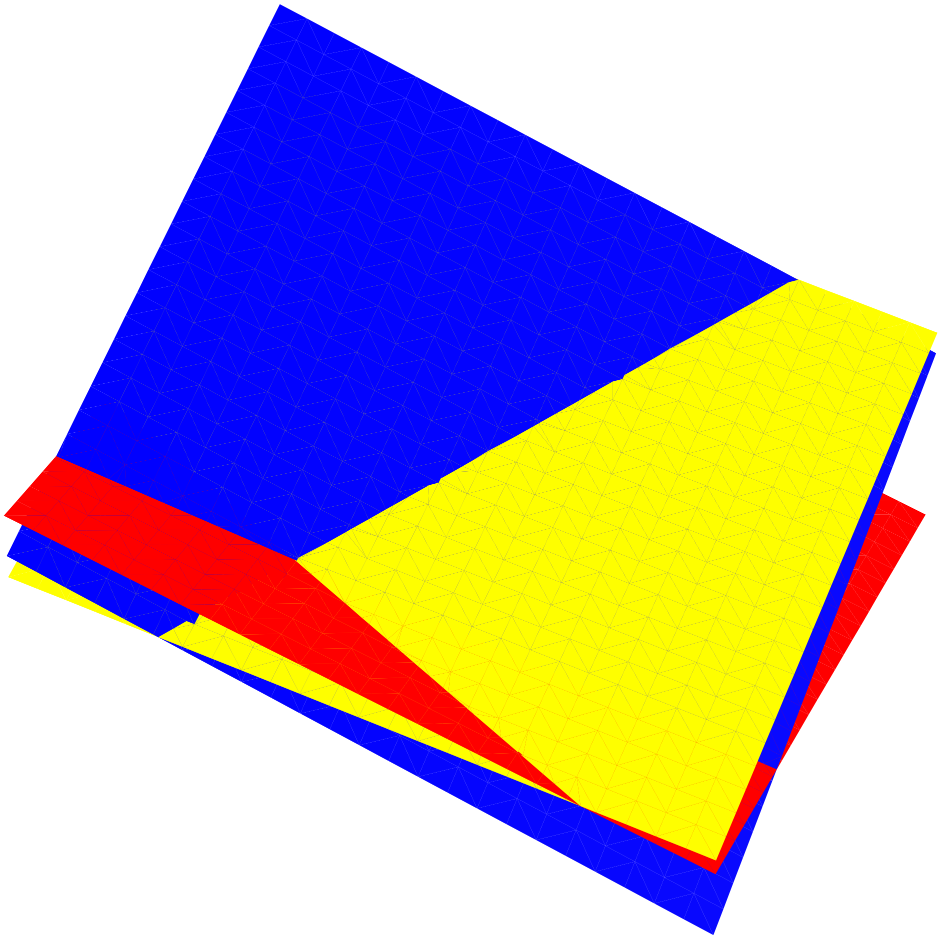


One-Point Intersection

$$x - 2y - 3z = 4$$
$$2x + 3y + 4z = 4$$
$$3x - 4y - 7z = 10$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 3$$

One-Point Intersection



One-Line Intersection

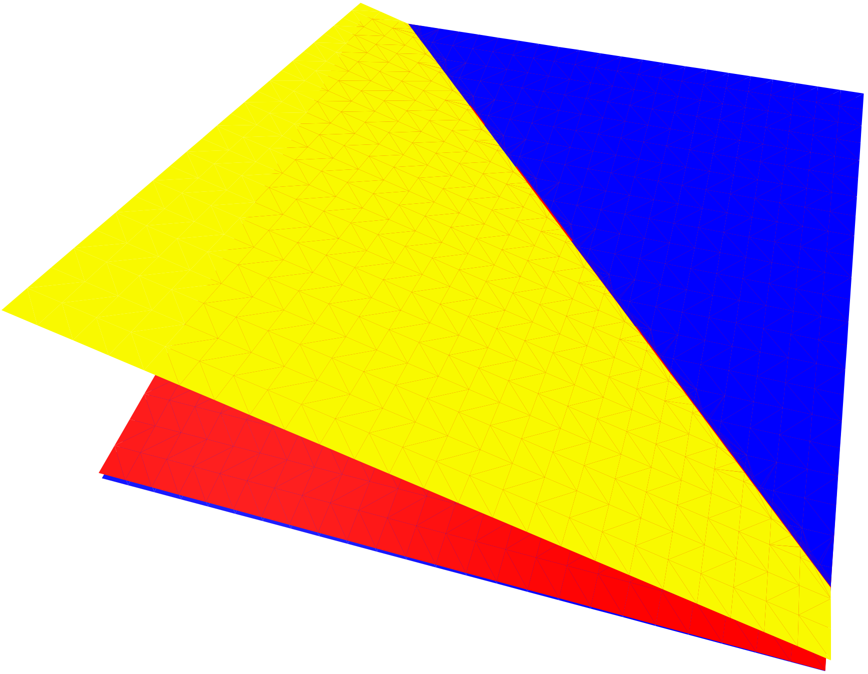
$$x - 2y - 3z = 4$$

$$2x + 3y + z = 1$$

$$3x - 4y - 7z = 10$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 2 < 3$$

One-Line Intersection

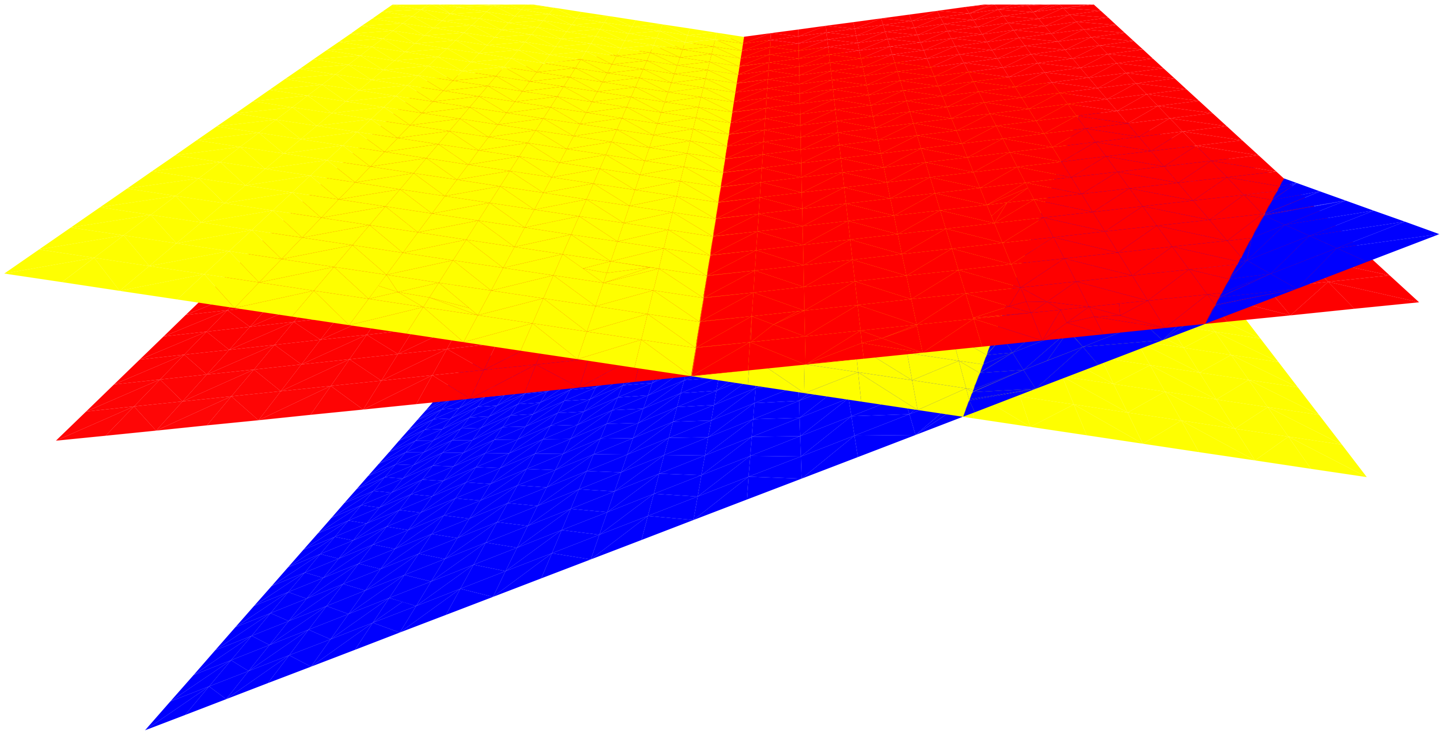


Three-Lines Intersection

$$\boxed{\begin{aligned} 3x + 6y + 9z &= 60 \\ 2x - 4y + 6z &= 40 \\ 2x + 7y - 3z &= 13 \end{aligned}}$$

$$\operatorname{rank} A = 2 < \operatorname{rank} \tilde{A} = 3$$

Three-Lines Intersection



Parallel Two-Lines Intersection

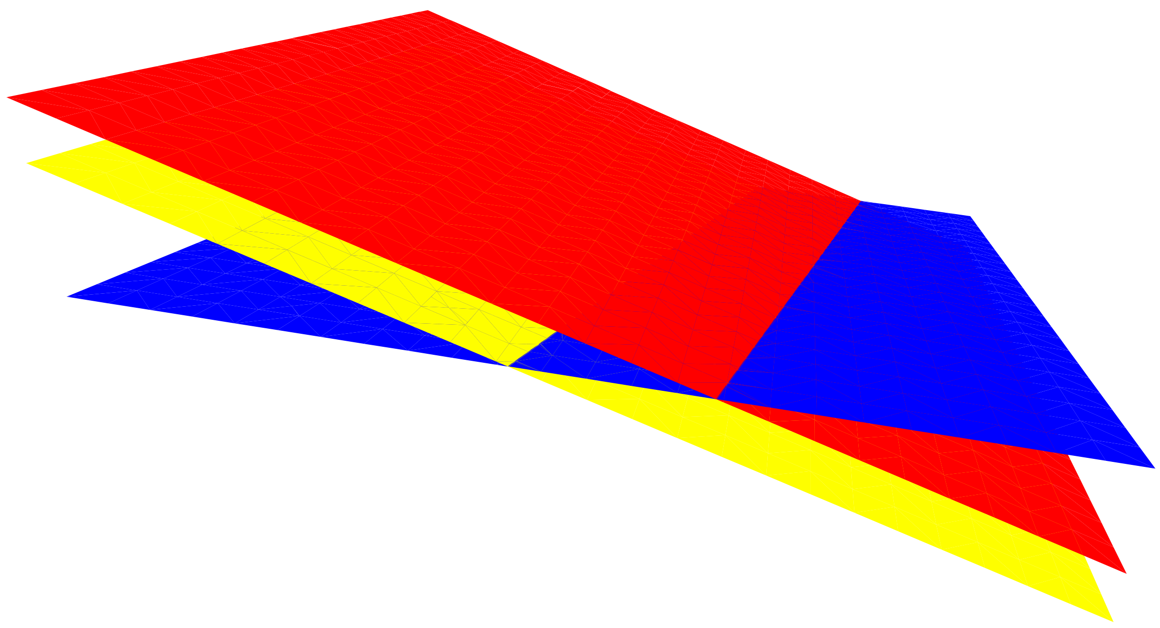
$$x - 2y - 3z = 1$$

$$x - 2y - 3z = 4$$

$$3x - 4y - 7z = 10$$

$$\text{rank } A = 2 < \text{rank } \tilde{A} = 3$$

Parallel Two-Lines Intersection

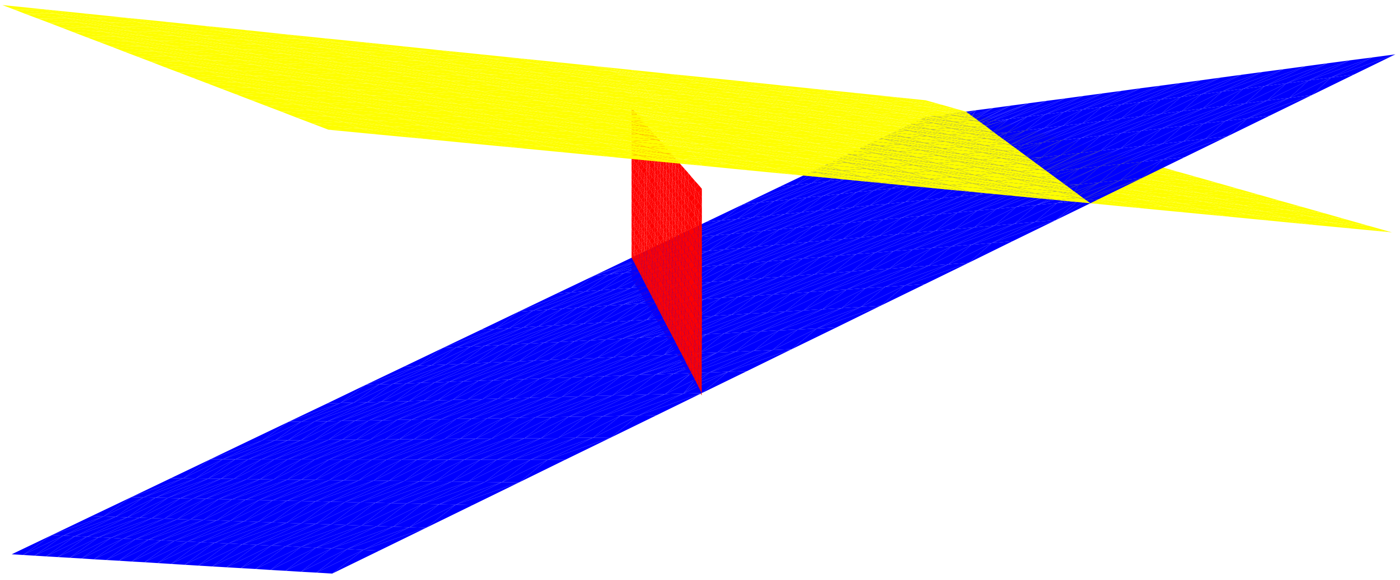


Parallel Three-Lines Intersection

$$
\boxed{\begin{aligned}
3x + 6y + 9z &= 60 \\
2x + 7y - 3z &= 13 \\
3x + 9y \phantom{{}+9z} &= 0
\end{aligned}}
$$

$$\operatorname{rank} A = 2 < \operatorname{rank} \tilde{A} = 3$$

Parallel Three-Lines Intersection



Parallel Two Planes

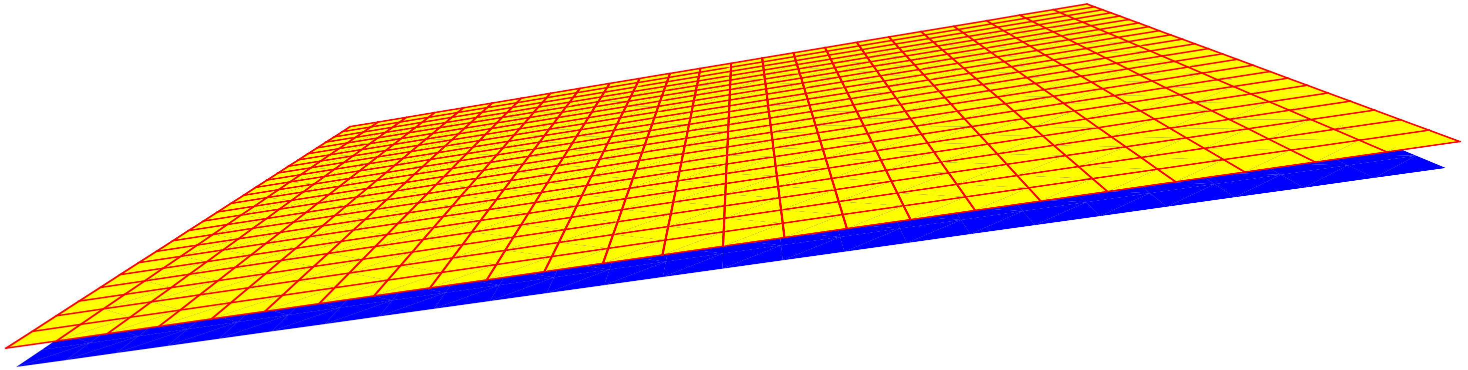
$$x - y + 3z = 1$$

$$3x - 3y + 9z = 3$$

$$x - y + 3z = 0$$

$$\text{rank}\, A = 1 < \text{rank}\, \tilde{A} = 2$$

Parallel Two Planes



Parallel Three Planes

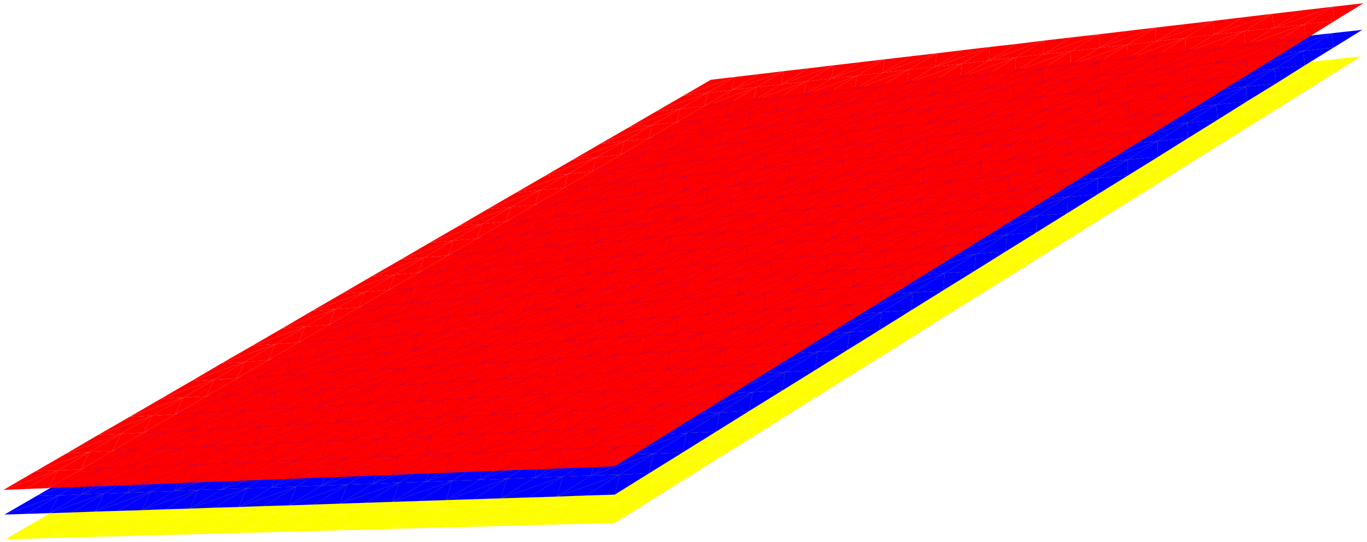
$$x - y + 3z = 1$$

$$x - y + 3z = 0$$

$$x - y + 3z = -1$$

$$\text{rank}\, A = 1 < \text{rank}\, \tilde{A} = 2$$

Parallel Three Planes

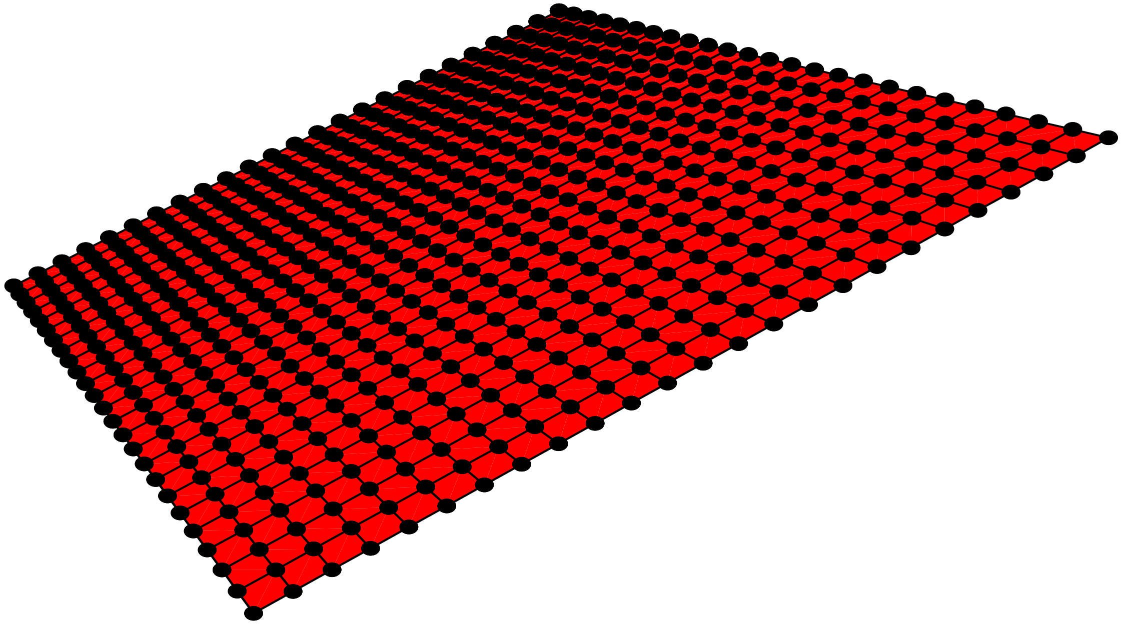


Superposed Three Planes

$$x - y + 3z = 1$$
$$3x - 3y + 9z = 3$$
$$2x - 2y + 6z = 2$$

$$\operatorname{rank} A = \operatorname{rank} \tilde{A} = 1 < 3$$

Superposed Three Planes



Jordan’s Theory

Marie Ennemond Camille Jordan

Jordan

- Marie Ennemond Camille Jordan (1838-1922)

French Mathematician

Jordan Canonical Form

of

Matrices

Jordan's Canonical Form (1)

$$\boxed{A\mathbf{x} = \mathbf{b}}$$

Change of Bases $\downarrow$ $\uparrow$ **Original Form**

$$\boxed{AP\mathbf{y} = \mathbf{b},\ \mathbf{x} = P\mathbf{y}\ \Rightarrow\ (P^{-1}AP)\mathbf{y} = P^{-1}\mathbf{b}}$$

Jordan's Canonical Form (2)

$$\boxed{A\mathbf{x} = \mathbf{b}}$$

Change of Bases $\downarrow$ $\uparrow$ Original Form

$$\boxed{J\mathbf{y} = \mathbf{c}, \quad J = P^{-1}AP, \mathbf{c} = P^{-1}\mathbf{b}}$$

Classification of System of Linear Equations

General Form (n=3)

$$\boxed{\begin{aligned} ax + by + cz &= \alpha \\ dx + ey + fz &= \beta \\ gx + hy + iz &= \gamma \end{aligned}}$$

Matrix Representation (1)

$$\boxed{A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}$$

Matrix Representation (2)

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Matrix Representation (3)

$$\boxed{\begin{aligned} ax + by + cz &= \alpha \\ dx + ey + fz &= \beta \\ gx + hy + iz &= \gamma \end{aligned}}$$

$$\Downarrow$$

$$\boxed{A\mathbf{x} = \mathbf{b}}$$

Patterns of Simultaneous Linear Equations

$$\boxed{\begin{aligned} \lambda x &= \alpha \\ \mu y &= \beta \\ \nu z &= \gamma \end{aligned}}$$

$$\boxed{\begin{aligned} \lambda x + y &= \alpha \\ \lambda y &= \beta \\ \nu z &= \gamma \end{aligned}}$$

$$\boxed{\begin{aligned} \lambda x + y &= \alpha \\ \lambda y + z &= \beta \\ \nu z &= \gamma \end{aligned}}$$

Matrix Representation

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Jordan Canonical Form 1

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$J = P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

Pattern 1

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$3 = 1 + 1 + 1$$

Simultaneous Linear Equation 1

$$\boxed{\begin{aligned} \lambda \mathbf{x} &= \alpha \\ \mu \mathbf{y} &= \beta \\ \nu \mathbf{z} &= \gamma \end{aligned}}$$

Jordan Canonical Form 2

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$J = P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

Pattern 2

$$\boxed{\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}}$$

$$\boxed{3 = 2 + 1}$$

Simultaneous Linear Equation 2

$$\lambda x + y = \alpha$$

$$\lambda y = \beta$$

$$\nu z = \gamma$$

Jordan Canonical Form 3

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$J = P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

Pattern 3

$$\boxed{\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}}$$

$$\boxed{3 = 3}$$

Simultaneous Linear Equation 3

$$\boxed{\begin{aligned} \lambda x + y &= \alpha \\ \lambda y + z &= \beta \\ \nu z &= \gamma \end{aligned}}$$

Linea Algebra and Differential Equations

Linear Case

Second-Order Case

$$\begin{cases} u''(t) + 2bu'(t) + cu(t) = 0, \\ u(0) = u_0, \\ u'(0) = u_1 \end{cases}$$

General Solutions

General Solution（1）

$$\boxed{D/4 = b^2 - c > 0}$$

$$u(t) = e^{-bt}\left(Ae^{t\sqrt{b^2-c}} + Be^{-t\sqrt{b^2-c}} \right)$$

A, B : **Constants**

Example

$$\begin{cases} x''(t) - x(t) = t \\ x(0) = x'(0) = 0 \end{cases}$$

Solution : $x(t) = \dfrac{1}{2}(e^{t} - e^{-t}) - t$

General Solution（2）

$$\boxed{D/4 = b^2 - c < 0}$$

$$u(t) = e^{-bt}\left(A\cos\sqrt{c-b^2}\,t + B\sin\sqrt{c-b^2}\,t\right)$$

A, B : **Constants**

Euler's Formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

General Solution（3）

$$D/4 = b^2 - c = 0$$

$$u(t) = e^{-bt}\left(At + B\right)$$

$A, B:$ **Constants**

Exponential Matrix

Exponential Function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$+ \frac{x^n}{n!} + \cdots$$

Main Idea

$$u''(t) + 2bu'(t) + cu(t) = 0$$

$$u''(t) + 2bu'(t) + cu(t) = 0$$

Matrix Representation

Original Form

$$\frac{dU(t)}{dt} = AU(t) \implies \textbf{Calculation of } e^{tA}$$

Solution (1)

$$\boxed{\begin{cases} u_1(t) = u(t), \\ u_2(t) = u'(t) \end{cases}}$$

$$\boxed{\begin{cases} u_1'(t) = u'(t) = u_2(t), \\ u_2'(t) = u''(t) = -2bu'(t) - cu(t) \\ \qquad\qquad = -2bu_2(t) - cu_1(t) \end{cases}}$$

Solution (2)

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}$$

Solution (3)

$$U(t)=\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$A=\begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\begin{cases} \dfrac{d}{dt}U(t)=AU(t), \\ U(0)=U_0 \end{cases}$$

Solution (4)

$$\boxed{U(t) = e^{tA} U_0}$$

$$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots$$

(**Exponential Matrix**)

Example
of
Exponential Matrices

Simple Eigenvalue Case

Calculation (1)

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\left|\lambda I - A\right| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c$$

Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c \neq 0$$

$$\begin{cases} \lambda_1 = -b + \sqrt{b^2 - c}, \\ \lambda_2 = -b - \sqrt{b^2 - c} \end{cases}$$

Calculation (3)

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -b+\sqrt{b^2-c} & -b-\sqrt{b^2-c} \end{pmatrix}$$

$P^{-1}AP = \Lambda$ **(Diagonal)**

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -b+\sqrt{b^2-c} & 0 \\ 0 & -b-\sqrt{b^2-c} \end{pmatrix}$$

Calculation (4)

$$P^{-1}e^{tA}P$$

$$= P^{-1}\left(I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right)P$$

$$= P^{-1}P + t\left(P^{-1}AP\right) + \frac{t^2}{2!}\left(P^{-1}AP\right)\left(P^{-1}AP\right) + \cdots +$$

$$+ \frac{t^n}{n!}\underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}_{n-\text{times}} + \cdots$$

$$= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= e^{t\Lambda}$$

Calculation (5)

$$
\begin{aligned}
e^{t\Lambda} &= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \lambda_1{}^2 & 0 \\ 0 & \lambda_2{}^2 \end{pmatrix} + \cdots \\
&\quad + \frac{t^n}{n!}\begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix} + \cdots \\
&= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}
\end{aligned}
$$

Calculation (6)

$$e^{tA} = Pe^{t\Lambda}P^{-1}$$

$$= \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1\lambda_2\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

Calculation (7)

$$\textbf{Case}: D/4 = b^2 - c \neq 0$$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1 \lambda_2 \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

Double Eigenvalue Case

Jordan's Canonical Form

$$P^{-1}AP = \Lambda \quad \textbf{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Calculation (1)

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}}$$

$$\boxed{\left|\lambda I - A\right| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c}$$

Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c = 0$$

$$\lambda = -b \quad \textbf{(Double Root)}$$

$$P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

Calculation (3)

$$P^{-1}AP = \Lambda \quad \textbf{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -b & 1 \\ 0 & -b \end{pmatrix}$$

Calculation (4)

$$
\begin{aligned}
&P^{-1}e^{tA}P \\
&= P^{-1}\left(I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right) P \\
&= P^{-1}P + t\left(P^{-1}AP\right) + \frac{t^2}{2!}\left(P^{-1}AP\right)\left(P^{-1}AP\right) + \cdots + \\
&\qquad + \frac{t^n}{n!}\underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}_{n-\text{times}} + \cdots \\
&= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots \\
&= e^{t\Lambda}
\end{aligned}
$$

Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!}\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Calculation (6)

$$e^{tA} = Pe^{t\Lambda}P^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} - \lambda te^{\lambda t} & te^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1)e^{\lambda t} \end{pmatrix}$$

Calculation (7)

$$\textbf{Case}: D/4 = b^2 - c = 0$$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} - \lambda t e^{\lambda t} & t e^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1) e^{\lambda t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

Vector Analysis

Inner Product (1)

$$\mathbf{a} = \left(a_1, a_2, a_3\right), \mathbf{b} = (b_1, b_2, b_3)$$

$$\Rightarrow$$

$$(\mathbf{a}, \mathbf{b}) = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\theta$$

$$\|\mathbf{a}\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$$

$$\|\mathbf{b}\| = \sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2}$$

Inner Product (2)

$$\mathbf{a} = \left(a_1, a_2, \cdots, a_n \right), \mathbf{b} = (b_1, b_2, \cdots, b_n)$$

$$\Rightarrow$$

$$(\mathbf{a}, \mathbf{b}) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Cross Product (1)

$$\mathbf{a} = \left(a_1, a_2, a_3\right), \ \ \mathbf{b} = \left(b_1, b_2, b_3\right)$$

$$\Rightarrow$$

$$\mathbf{a} \times \mathbf{b}$$

$$= \left(\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right)$$

$$= \left(a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \right)$$

Cross Product (2)

$$\mathbf{a} = \left(a_1, a_2, a_3\right), \;\; \mathbf{b} = \left(b_1, b_2, b_3\right)$$

$$\Rightarrow$$

$$\left\|\mathbf{a} \times \mathbf{b}\right\| = \left\|\mathbf{a}\right\| \cdot \left\|\mathbf{b}\right\| \sin\theta$$

Gradient

$$\boxed{\mathrm{grad}\ f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)}$$

Rotation (1)

$$\boxed{\begin{aligned}&\operatorname{rot}(f, g, h)\\&=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}, \frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}, \frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\end{aligned}}$$

Rotation (2)

$$\nabla = \left(\partial_x, \partial_y, \partial_z\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\mathbf{F} = (f, g, h)$$

$$\Longrightarrow$$

$$\mathrm{rot}\,(f, g, h) = \nabla \times \mathbf{F}$$

Rotation (3)

$$\mathrm{rot}(f, g, h) = \nabla \times \mathbf{F}$$

$$= \left(\begin{vmatrix} \partial_y & \partial_z \\ g & h \end{vmatrix}, \begin{vmatrix} \partial_z & \partial_x \\ h & f \end{vmatrix}, \begin{vmatrix} \partial_x & \partial_y \\ f & g \end{vmatrix} \right)$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)$$

Divergence (1)

$$\boxed{\operatorname{div}(f,g,h)=\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}}$$

Divergence (2)

$$\nabla = \left(\partial_x, \partial_y, \partial_z\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\mathbf{F} = (f, g, h)$$

$$\Longrightarrow$$

$$\operatorname{div}(f, g, h) = \nabla \bullet \mathbf{F}$$

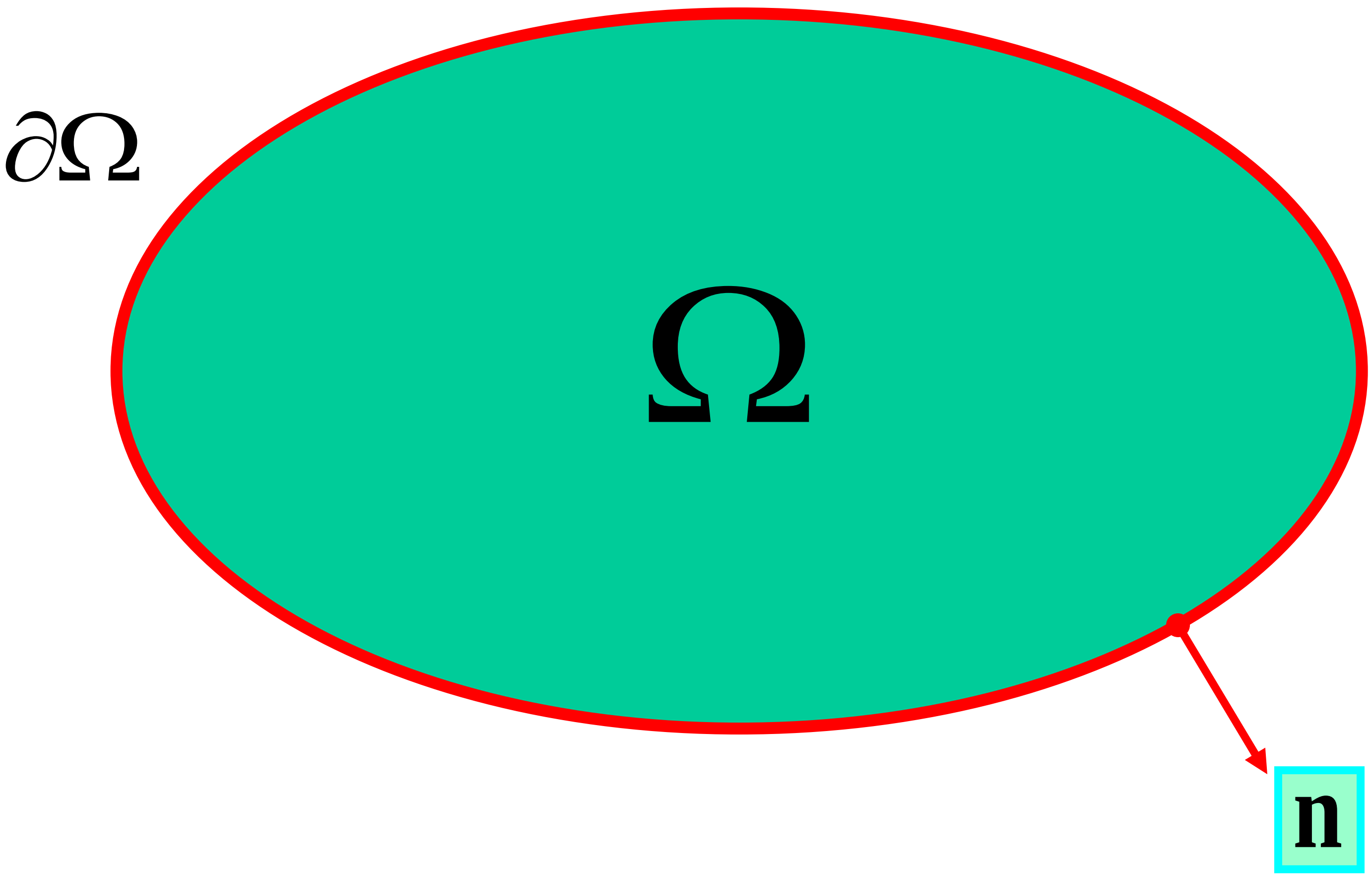
Well-known Formulas

$$\boxed{\mathrm{rot} \circ \mathrm{grad}\ f = 0}$$

$$\boxed{\mathrm{div} \circ \mathrm{rot}\ \mathbf{v} = 0}$$

Green's Theorem

2-dimensional Domain



Green's Theorem (1)

$$\iint_{\Omega} \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \right) dxdy$$

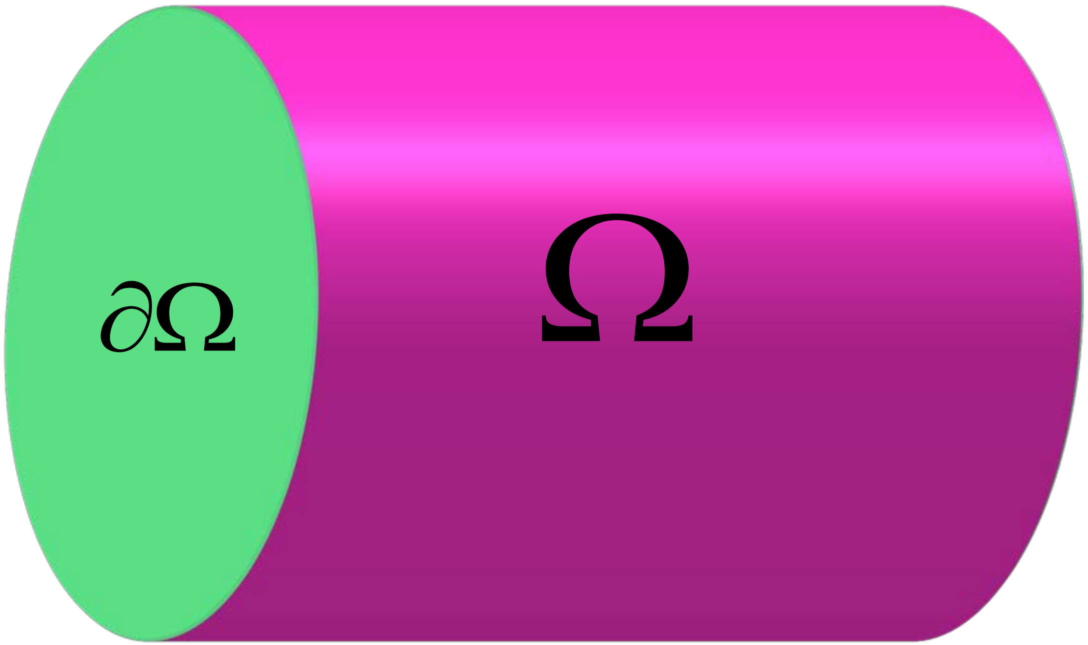
$$= \int_{\partial\Omega} f dy + g dx$$

Green's Theorem (2)

$$\boxed{\begin{gathered}\iint_{\Omega} \operatorname{div}\mathbf{F}\, dv = \int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\, ds \\ \mathbf{F} = (f, g)\end{gathered}}$$

Gauss' Divergence Theorem

3-dimensional Domain



Gauss' Divergence Theorem (1)

$$\boxed{\begin{aligned}&\iiint_{\Omega}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)dxdydz\\&=\iint_{\partial\Omega} fdydz+gdzdx+hdxdy\end{aligned}}$$

Gauss' Divergence Theorem (2)

$$\iiint_D \mathrm{div}\mathbf{F}\, dV = \iint_{\partial D} \mathbf{F}\cdot\mathbf{n}\, dS$$

$$\mathbf{F} = (f, g, h)$$

Application to Electro-magnetism

Gauss' Theorem (Magnetic Field)

$$\iint_{\partial D} \mathbf{B}(x) \cdot \mathbf{n}\, dS = 0$$

$$\mathbf{B}(x) = \textbf{Magnetrostatics}$$

Gauss' Theorem (Electric Field)

$$\iint_{\partial D} E(x)\cdot\mathbf{n}\,dS = \frac{1}{\varepsilon_0}\iiint_D \rho(x)\,dx$$

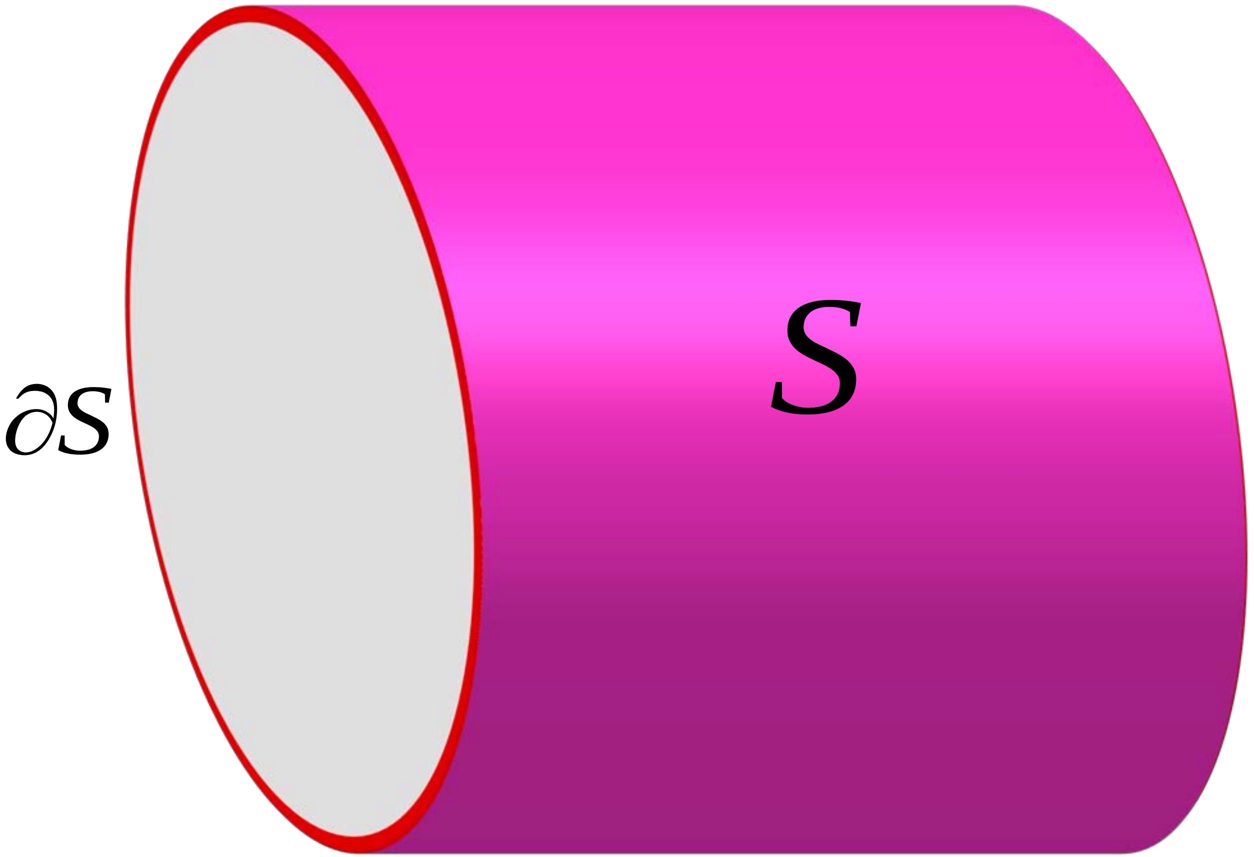
$E(x) =$ **Electrostatic Field**

$\rho(x) =$ **Electric Density**

$\varepsilon_0 =$ **Inductive Capacity in Free Space**

Stokes' Theorem

Surface



Stokes' Theorem (1)

$$\iint_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dydz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dzdx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$= \int_{\partial S} fdx + gdy + hdz$$

Stokes' Theorem (2)

$$\iint_S \operatorname{rot} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s},$$

$$\mathbf{F} = (f, g, h)$$

Application to Electro-magnetism

Faraday's Law

$$\boxed{\begin{gathered} -\frac{d}{dt}\left(\iint_S \mathbf{B}(x,t)\cdot\mathbf{n}\,dS\right) = \int_{\partial S} \mathbf{E}(x,t)\cdot d\mathbf{r}\,, \\ d\mathbf{r} = (dx, dy, dz) \end{gathered}}$$

END